# Chapter 20

# Entanglement

## 20.1 Bohr vs. Einstein

The story of the birth of quantum mechanics could not be a more interesting tale. It started with the mounting evidence that our view of atomic phenomena had to change. Niels Bohr felt deeply about this. Earlier in the 20th century Max Plank and Albert Einstein had defined the quantum of light to successfully explain the radiation of black body and the photoelectric effect. Then Bohr goes to work with Ernest Rutherford when Rutherford puts forth the model of the atomic nucleus. This led Bohr to develop an atomic model that explaned the line spectra of hydrogen. But Bohr's model had limitations, and did not explain the mounting evidence for the duality of light and matter. Recognizing that the description of atomic phenomena required a new approach, Bohr headed in Copenhagen the Institute for Atomic Studies. This institute became the birthplace of quantum mechanics, harboring important contributors to the theory, such as Werner Heisenberg, Max Born, Wolfgang Pauli, Paul Dirac and others. The theories of quantum mechanics came to fruition in the years 1925-1926, with important contributions by Heisenberg and Erwin Schrodinger, in Germany.

Bohr stated that one of the cornerstones of the new theory was the wave-particle duality. He maintained that it rested on the *principle of complementarity*, that the wave and particle pictures are "complementary" to each other. That is, they are mutually exclusive, yet essential for a complete description of quantum phenomena. In the case of a double-slit experiments, if we choose not to inquire which arm the photon goes through (wave aspect) we find interference. If we determine through which arm the photon goes (particle aspect) we see no interference.

Quantum mechanics got a major exposure at the Solvay conference of 1927. The conference was famous for Einstein's challenges to the quantum theory, which were directed at Bohr. Figure 20.1 shows a photograph of the participants, which shows that it was quite a distinguished group.

Einstein's challenges came in terms of his famous *gedanken* or "thought" experiments. One of Einstein's challenges concerned a particle or photon going through a pair of slits. In this famous thought experiment depicted in Fig. 20.2, a plate with a pair of slits was on rollers such that we could measure the kick that a photon gave to the slit. This would lead us to determine the slit that the photon took in going to the screen where the interference pattern is formed. It

Figure 20.1: Participants of the 1927 Solvay conference. Front Row: I. Langmuir, M. Planck, Mme. Curie, H.A. Lorentz, A. Einstein, P. Langevin, Ch. E. Guye, C.T.R. Wilson, O.W. Richardson Middle Row: P. Debye, M. Knudsen, W.L. Bragg, H.A. Kramers, P.A.M. Dirac, A.H. Compton, L. de Broglie, M. Born, N. Bohr Back Row: A. Piccard, E. Henriot, P. Ehrenfest, Ed. Herzen, Th. De Donder, E. Schrodinger, E. Verschaffelt, W. Pauli, W. Heisenberg, R.H. Fowler, L. Brillouin. Photo AIP.

would violate complementarity.

**Exercise 1** For simplicity in our calculations assume that the photons hit the pair of slits coming parallel from the left. They come with a momentum $p$.

1. Find the tangent of the angle of deflection 1 of the photon if it goes through the top slit in terms of $y$, $d$ and $L$. (Ans: $tan\theta_1 = (y - d/2)/L$)

2. Do the same for the angle of deflection 2 of the photon going through the bottom slit. (Ans: $tan\theta_2 = (y + d/2)/L$)

3. The photon gets a momentum kick that adds a transverse component to the momentum of the photon $p_y$ without affecting the longitudinal momentum $p$. To tell which slit the photon takes we need to be able to measure a momentum kick smaller than the difference in the momentum kicks. Find an expression for the difference in momentum kicks $(p_{2y} - p_{1y})$. To simplify things let's further assume that the screen on the right is far enough away that the small-angle approximation is valid. (Ans: $\Delta p_y = pd/L$)

4. Bohr responded the next day by arguing that allowing the slits to move for determining the slit that the photon goes through gives uncontrollable shifts to the slits, destroying the interference pattern. Une could see this via the uncertainty principle. Let's go over his argument.

   (a) To distinguish the kicks from each other we need an uncertainty no less than $\Delta p_y = p_{2y} - p_{1y}$. An uncertainty $\Delta p_y$ implies that we would have an uncertainty $\Delta y \geq h/\Delta p_y$. That is, the uncertainty

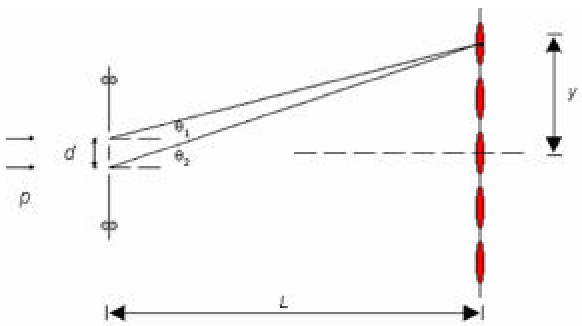


Figure 20.2: Diagram of Einstein's thought experiment.

principle reveals that the momentum kick on the slits will result in an uncertainty in the position of the slits greater than $h/\Delta p_y$. Find an expression for this value in terms of the variables of the apparatus and the wavelength of the photon. (Ans: $h/\Delta p_y = L\lambda/d$)

(b) Regarding the interference pattern:

i. Show that in the small-angle approximation, the position of the $n$-th maximum is $y_n = nL/d$.

ii. Using the previous result find the difference in positions of adjacent maxima $\delta y = y_n - y_{n-1}$. (Ans: $\delta y = \lambda L/d$)

If we put the uncertainty relation for $\Delta y$ in terms of $\delta y$ we get $\Delta y \geq \delta y$. Thus, we can understand that the pattern gets blurred because the uncertainty in the slit position (due to uncontrollable kicks) is greater than the separation of maxima of the interference pattern.

## 20.2 Energy-time uncertainty

Let's consider another consequence of the uncertainty relation as it relates to the photon. A photon can be viewed as a "wavepacket." This is a photon with a wavelength that is not perfectly defined. It has a spread $\Delta\lambda$.

**Exercise 2** Consider the following:

1. The spread $\Delta\lambda$ implies a spread $\Delta p$. It can be shown that when two variables are inversely proportional, like $p = h/\lambda$, their relative uncertainties are equal: $\Delta p/p = \Delta\lambda/\lambda$ Find an expression for $\Delta p$ in terms of $\Delta\lambda$ and $\lambda$. (Ans: $h\Delta\lambda/\lambda^2$)

2. Use the uncertainty principle to find the uncertainty in the position of the photon $\Delta x$, or the length of the wavepacket. This quantity is also known as the "coherence length." (Ans: $\lambda^2/(2\pi\Delta\lambda)$)

There are important consequences to the finite length of the wavepacket as it relates to the distinguishability of a photon going through an interferometer. For example, suppose that we send a 916-nm photon with a wavelength spread $\Delta\lambda = 0.1$ nm to an Mach-Zehnder interferometer. If the arms are nearly the same the paths are indistinguishable. If we increase the length of the arms so that they differ by more than the length of the wavepacket, then the paths become distinguishable. This is because a photon going through the shorter arm will arrive to the detector noticeably earlier than going through the longer arm.

**Exercise 3** What is the length of the wavepacket? (Ans. 1.33 mm)

**Exercise 4** What is the temporal spread $\Delta t$ of the photon? (It is traveling with a speed $c$.) (Ans: 4.45 ps)

If we use $E = pc$, $\Delta x = c\Delta t$ and the position-momentum uncertainty principle $\Delta x \Delta p \geq \hbar$ we get the time-energy uncertainty relation:

$$\Delta E \Delta t \geq \hbar \tag{20.1}$$

.

## 20.3 Delayed Choice

The thought experiment of Einstein on the moving slits is one example of the blurring of the interference pattern by the action of measurement. Measurement disturbs the system in an uncontrollable way that makes the interference pattern disappear. Bohr justified this using the uncertainty principle.

If we focus on the quantum eraser experiment, we find that the action of rotating the polarization of the light makes the path information available. This is done by entangling the information with the measurement apparatus. When we put the polarizer after the interferometer we are selecting out the path information. This is done not by disturbing the system but by making the path information unavailable. When the path information is present there is no interference. When we choose not to have the path information available we regain the interference. This effect of manipulatine the information is known as "delayed choice."

## 20.4 The EPR paradox

Back to Einstein and Bohr, there was one more subsequent conference, also at Solvay, in 1930, where Einstein challenged Bohr with another *gedankenexperiment*. Similarly to the 1027 conference Bohr was able to refute this thought experiment too. We will not discuss it here. However, in 1935 Einstein, Podolski and Rosen published in *Physical Review* a now famous article presenting yet another thought experiment. However, this time Bohr was not able to demonstrate the inadequacy of this one. This thought experiment became known as the "EPR" paradox. It presented a situation where quantum mechanics and realistic theories would predict distinct views. However, there was no apparent way to prove either way. At least not until 1964, when John Bell published the basis for a test that distinguished the two views. This test has become known as *Bell's inequalities*. In the last twenty years numerous experiments have verified a violation of these inequalities. In all cases the results of the experiments favored quantum mechanics.

## 20.5 Entanglement

The EPR paradox relies on entangled states of two particles. Before discussing the paradox lets discuss a modern realization of entangled states. These states can be created using correlated particles. There are many ways of producing correlated particles. One way involves the production of photon pairs in the process of *parametric down conversion*. In this process, a photon is incident on a nonlinear crystal producing pairs of photons. This is shown in Fig. 20.3. We label the incident photon the "pump," and the two down-converted photons as 1 and 2.

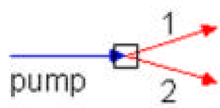


Figure 20.3: Diagram of spontaneous parametric down conversion.

These photons are correlated in the following ways:

- Energy correlation. The energies of the down-converted photons add up to the energy to the pump photon: $E_{pump} = E_1 + E_2$. In our experiments (see quantum eraser lab) we have a pump photon with a wavelength of 457.9 nm. While down-conversion produces a large range of photon pairs with complementary energies, for simplicity we work with the "degenerate" photons (i.e., that have the same energy), and thus a wavelength of 915.8 nm.
- Momentum correlation. The momentum of the photons is conserved: $\mathbf{p}_{pump} = \mathbf{p}_1 + \mathbf{p}_2$. Since we work with the photons that have the same energy, they come out at the same angle relative to the incident beam. In the lab this angle is 3 degrees.
- Time correlation. The photons are produced simultaneously. We use this fact in the detection of the down-converted photons: we detect coincidences. This way we are able to separate down-converted photons from all other stray photons.
- Polarization correlation. In our experiments the down-converted photons coming from a single crystal have the same polarization.

While our photons are correlated, they are not necessarily entangled. Suppose that we have two photons that are vertically polarized. We can analyze the polarization correlation by detecting the photons after they pass through polarizers. Figure 20.4 shows a schematic of the experiment. The polarizers $P_1$ and $P_2$ have their transmission axes oriented at angles $\theta_1$ and $\theta_2$ relative to the vertical, respectively. The circle represents the down-conversion source. Because the photons are correlated the probability amplitude that both photons get detected is given by

$$\phi_{VV} = \cos\theta_1 \cos\theta_2. \tag{20.2}$$

The probability is, as always, the square of the probability amplitude. The state referred to by equation 20.2 is a "product state." It is the same as a classical source, because we know the polarization of the photons all along.

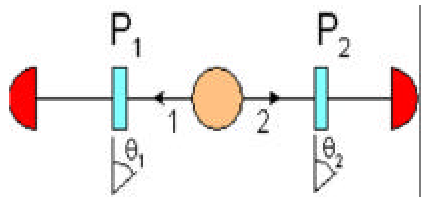


Figure 20.4: Schematic of the experiment to do polarization measurements on correlated photons.

**Exercise 5** Show that the probability amplitude when both photons are horizontal is given by

$$\phi_{HH} = \sin\theta_1 \sin\theta_2. \tag{20.3}$$

To be entangled the photons have to be in a superposition of possibilities. We can create this situation using a clever trick invented recently and shown in Fig. 20.5. We get two crystals: a first one produces photons that are horizon-

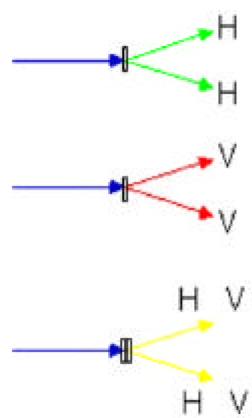


Figure 20.5: Diagram of the method of producing polarization entangled states. Two thin crystals produce correlated photons that are horizontally polarized (top), and vertically polarized (middle). They are put together in a new arrangement (bottom) that produces photons that are entangled: in a superposition of being horizontal and vertical.

tally polarized and a second one produces pairs that are vertically polarized. By putting the two crystals together we create a new situation that produces photons in the two polarizations. Because the width of the crystals is much smaller than the width of the beam there is no way to tell in which crystal the photons got created. As a consequence the photon pairs are in a superposition of being both horizontal and vertical.

If we repeat the experiment of Fig. 20.4, we find that because the photons are entangled the probability amplitude of detecting the photons is

$$\phi_E = \frac{1}{\sqrt{2}}(\phi_{HH} + \phi_{VV}). \tag{20.4}$$

While Eq. 20.4 refers to the probability amplitude of detection, let's refer to this entangled state as the state HH+VV. If we replace the expressions from Eqs. 20.2 and 20.3 we get

$$\phi_E = \frac{1}{\sqrt{2}}(\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_1). \tag{20.5}$$

After simplifying we get

$$\phi_E = \frac{1}{\sqrt{2}}\cos(\theta_1 - \theta_2). \tag{20.6}$$

**Exercise 6** The photons in the entangled state HH+VV go through the two polarizers of Fig. 20.4. Fill the table below:

| $\theta_1$ | $\theta_2$ | $\phi_E$ | $P_E = \lvert\phi_E\rvert^2$ |
|---|---|---|---|
| 0° | 0° | | |
| 0° | 90° | | |
| 90° | 90° | | |
| +45° | +45° | | |
| +45° | -45° | | |

Note that the entangled state is not a classical state. By this we mean that we cannot compute the probabilities and later add them. We first add the amplitudes and then compute the probabilities.

Notice also something peculiar about Eq. 20.6. The probability is a maximum when $\theta_1 = \theta_2$. This means that the entangled state of Eq. 20.6 is much more that what it seems. The HH+VV entangled state is not jut the superposition of photons polarized horizontally with photons polarized vertically. It is a state of photons that are parallel to each other *regardless of the orientation*. Moreover, quantum mechanics implies that the photon pair is in a superposition of all parallel orientations until a measurement is made. Once we do the measurement on both, then they have a definite polarization state and they are no longer entangled.

From the above discussion then if we alow one photon to pass through a polarizer, then we immediately know the polarization of the partner photon, *wherever it is*! Einstein could not believe this, and called it: *spukhafte fernwirkun*, or "spooky action at a distance."

This strange property of quantum systems is exploited today in *quantum cryptography*, a technology to establish secure communications. In quantum communications we refer to the sender of information as "Alice," and the receiver as "Bob." Consider the situation shown in the diagram of Fig. 20.6. Alice has an entangled source and takes one of the photons and passes it through a polarizer and detector. The other photon is sent to Bob, who is a long distance away.

**Exercise 7** Alice in Fig. 20.6 sends a photon to Bob that is entangled with hers.

1. If she detects a photon when her polarizer is horizontal, what is the probability that Bob will detect a photon if his polarizer is

   (a) horizontal,

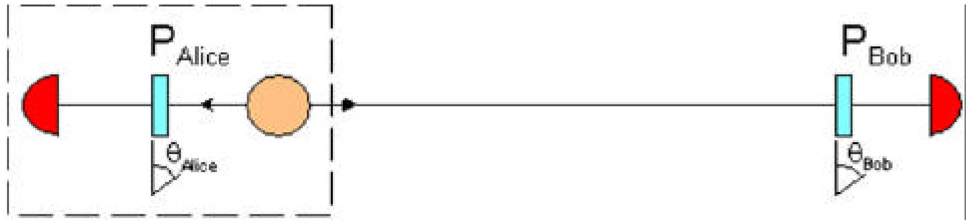


Figure 20.6: Diagram of the method of doing secure communications using entangled states.

(b) vertical,

(c) +45°,

(d) -45°

(Ans: 1, 0, 1/2, 1/2)

2. Alice detects a sequence of 8 photons with her polarizer set to the orientations shown in the table below. In turn, Bob independently detect the partner photons with his polarizer set to the values shown in the table.

| $\theta_{Alice}$ | $\theta_{Bob}$ | Agreement? |
|---|---|---|
| horizontal | vertical | |
| vertical | vertical | |
| vertical | +45° | |
| +45° | -45° | |
| -45° | -45° | |
| vertical | vertical | |
| +45° | vertical | |
| +45° | horizontal | |

Later Bob and Alice compare the settings of their polarizers. For which cases will they agree with certainty that they got the photon partners?

The interesting aspect of this type of communication is that an eavesdropper, "Eve," has no choice but to absorb the photon on its way to Bob in order to intercept the communication,and re-emit one. However, this action removes the correlation between the photon that Bob receives with Alice's.

One final remark about polarization-entangled states. The state HH+VV that we considered is only one of several possibilities. Another one is the state HV-VH, in which the photons are in a state where their polarizations are always perpendicular to each other, regardless of the orientation.

## 20.6 Problems

**Problem 1** At the 1927 Solvay meeting Bohr argued against Einstein's *gedanken-experiment* on the interference with a slits on rollers. Explain how Bohr used the uncertainty principle to refute Einstein.

**Problem 2** What equivalence did Bohr find between the uncertainty principle and diffraction? Explain the basis for this equivalence.

**Problem 3** A sodium atom in its excited state has a "lifetime" of 16 ns. One can view this lifetime as the uncertainty in the time at which a photon is emitted. The wavelength of the emitted photon is 590 nm.

1. When two variables $x$ and $y$ are inversely related as in $x = c/y$, where $c$ is a constant, Using derivatives find a relation between the changes in $x$ and $y$, $\Delta x$ and $\Delta y$. Note: the answer is NOT: $\Delta x = c/\Delta y$.

2. If the wavelength of a photon is uncertain by an amount $\Delta\lambda$, we can use the type of relation you found in (a) to find the corresponding uncertainty in the energy $\Delta E$. Find an expression for $\Delta E$ in terms of $\Delta\lambda$, $\lambda$ and any relevant constants.

3. Using the energy uncertainty relation $\Delta E \Delta t > \hbar$, find the uncertainty in the wavelength of the photon emitted by the sodium atom.

4. One can view the uncertainty in the time of emission of the photon as an inherent temporal spread of the photon: the photon is a "wave packet" with a temporal width $\Delta t$. If the photon travels at a speed $c$, what is the spatial length of the wave packet?

5. If the light entering a Mach-Zehnder interferometer is made of these photons. How is it that we can make the paths distinguishable without touching the polarization of the photon?

**Problem 4** Consider the experiment of Fig. 20.4.

1. If photon 1 is vertically polarized, and photon 2 is horizontally polarized, what is the probability amplitude $\phi_{HV}$ for joint detections it polarizers 1 and 2 have their transmission axes oriented at angles $\theta_1$ and $\theta_2$ relatinve t the vertical.

2. Now consider the case where the photons are in the entangled state HV-VH. In this case the probability amplitude for joint detections is

$$\phi = \frac{1}{\sqrt{2}}(\phi_{HV} - \phi_{VH}) \tag{20.7}$$

   Show that the probability for joint detections is

$$\phi = \frac{1}{\sqrt{2}}\sin(\theta_1 - \theta_2) \tag{20.8}$$

   The probability of joint detection is a maximum when the polarizations of the two photons are perpendicular to each other, regardless of their overall orientation.