Quantum Mechanics of Photon and Biphoton Experiments

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Contents

Chapter 1

Postulates

Let us begin with a few axiomatic principles of quantum mechanics.

1. A measurement on a system (e.g., a particle) gives a result, a real number in some units. Quantum mechanics makes predictions on any type of measurement via the state of the system. The physical state of the particle is represented by a state vector, denoted by the "ket" $|\psi\rangle$, and which carries all the physical information about the state that is allowed by quantum mechanics. The variable ψ is just a label representing the state.

2. If c is a complex number, $c|\psi\rangle$ is the same state as $|\psi\rangle$. The "bra" form of the state is the complex-conjugate of the state:
$$c^*\langle\psi| = (c|\psi\rangle)^*\,, \tag{1.1}$$
where the asterisk stands for complex conjugate.

3. The physical quantity that is measured is the observable. Let us call A the observable. It can represent any physical quantity. A photon carries spin or helicity, so the photon spin is an observable.

4. The results of the measurement of the observable are the eigenvalues. In the case of energy, for example, the eigenvalues are the possible energies of the system. These eigenvalues can be denoted by a_i, where $i = 1...N$. N is a positive integer representing the dimensionality of the space, which is given by the number of alternative outcomes of a measurement. In the case of the helicity of the photon, there are two observables $\pm\hbar$. In this case $N = 2$.

5. The physical state corresponding to each eigenvalue is the eigenvector. We denote the eigenvector associated to the eigenvalue a_i by $|a_i\rangle$, where we have labeled the state by its eigenvalue. If there are two states with the same eigenvalue, we would need to find an appropriate labeling that distinguishes between the two states. The space of eigenvectors is called the Hilbert space.

   In the case of the photon helicity, we can label the eigenstates by $|+\rangle$ and $|-\rangle$. The eigenstates of helicity are the states of left and right circular polarization, so we can alternatively label the states respectively by $|L\rangle$ and $|R\rangle$.

6. Eigenstates are mutually exclusive. Thus, they are orthonormal. We represent this property via the scalar product. The scalar product of two states $|\phi\rangle$ and $|\psi\rangle$ is given by

$$\langle\phi|\psi\rangle \tag{1.2}$$

   and which measure the amplitude or overlap of state $|\phi\rangle$ onto $|\psi\rangle$. This scalar product is similar to the inner product of vector algebra. In the case of eigenstates, orthonormality requires

$$\langle a_i|a_j\rangle = \delta_{ij}, \tag{1.3}$$

   where δ_{ij} is the Kroneker delta.

   In the case of states of photon helicity, orthonormality requires:

$$\langle R|R\rangle = 1 \tag{1.4}$$
$$\langle L|L\rangle = 1 \tag{1.5}$$

   or the amplitude of being oneself to be 1, and

$$\langle R|L\rangle = 0 \tag{1.6}$$
$$\langle L|R\rangle = 0 \tag{1.7}$$

   or the property of one eigenstate to be in the other is 0.

   Note also that

$$\langle a_j|a_i\rangle = \left(\langle a_i|a_j\rangle\right)^* . \tag{1.8}$$

7. The general state of the particle is in general a linear superposition of eigenstates

$$|\psi\rangle = \sum_i c_i |a_i\rangle, \tag{1.9}$$

where c_i is a complex number representing the probability amplitude. The probability amplitudes can also be expressed as

$$c_i = \langle a_i|\psi\rangle. \tag{1.10}$$

It is important to stress that superposition in quantum mechanics implies that the state of the system is simultaneously in the states of the superposition $|a_i\rangle$ with amplitudes c_i. The "+" sign plays a hugely significant role, beyond the algebraic one.

In the case of photons, the state of horizontal and vertical polarizations are superpositions of the helicity eigenstates:

$$|H\rangle = \frac{1}{\sqrt{2}}|R\rangle + \frac{1}{\sqrt{2}}|L\rangle \tag{1.11}$$

$$|V\rangle = \frac{i}{\sqrt{2}}|R\rangle - \frac{i}{\sqrt{2}}|L\rangle \tag{1.12}$$

Exercise 1 Show that $\langle H|V\rangle = 0$.

The state of the system can be rewritten as

$$|\psi\rangle = \sum_i |a_i\rangle\langle a_i|\psi\rangle. \tag{1.13}$$

Exercise 2 $|H\rangle$ and $|V\rangle$ are an orthonormal set of eigenstates. Use the above definition to express $|R\rangle$ and $|L\rangle$ in terms of $|H\rangle$ and $|V\rangle$.

8. The probability that a measurement of the observable yields an eigenvalue is the absolute-value squared of the probability amplitude corresponding to that eigenvalue. That is

$$\mathcal{P}_i = |a_i|^2 \tag{1.14}$$

$$\mathcal{P}_i = a_i^* a_i. \tag{1.15}$$

This relation can be expressed in several other forms:

$$\mathcal{P}_i = |\langle a_i|\psi\rangle|^2 \tag{1.16}$$

$$= |c_i|a_i\rangle|^2 . \tag{1.17}$$

Exercise 3 What is the probability of measuring a helicity of $+\hbar$ when the photon is in state $|V\rangle$?

9. The state vector is normalized:

$$\langle\psi|\psi\rangle = 1. \tag{1.18}$$

This then requires

$$\langle\psi|\psi\rangle = \left(\sum_i c_i^* \langle a_i|\right)\left(\sum_j c_j |a_j\rangle\right) \tag{1.19}$$

$$= \sum_i c_i^* c_i, \tag{1.20}$$

which implies

$$\sum_i c_i^* c_i = 1. \tag{1.21}$$

The latter implies that the sum of probabilities of all outcomes is 1:

$$\sum_i \mathcal{P}_i = 1. \tag{1.22}$$

10. The average value or expectation value of an observable is the sum of the result of all possible measurement outcomes times the probability of getting that outcome.

$$\langle A\rangle = \sum_i c_i^* c_i a_i. \tag{1.23}$$

Exercise 4 Find the expectation value of the spin of the photon $\langle S\rangle$ when the photon is in state $|H\rangle$.

11. A measurement of an observable yields an eigenvalue and projects the state of the system to the eigenvector corresponding to the eigenvalue found. Suppose that we have an initial state $|\psi\rangle$

$$|\psi\rangle = \sum_i c_i |a_i\rangle \tag{1.24}$$

and then we measure the observable A and get the eigenvalue a_j. The state of the system after the measurement is

$$|\psi\rangle_{\text{new}} = |a_j\rangle. \tag{1.25}$$

We note that the probability of obtaining eigenvalue a_j is $\mathcal{P}_j = c_j^* c_j$. The process of measurement is inherently indeterministic. That is, that before the measurement we have no idea of what the outcome will be. We only know the probabilities for each of the possible outcomes. This is in contrast with the determinism of classical physics, where the equation of motion in principle determines with certainty the future outcome of a measurement.

A photon linearly polarized along a direction that forms an angle θ relative to the horizontal can be expressed by a state $|\theta\rangle$ that is a linear superposition of states in the (H,V) basis

$$|\theta\rangle = \cos\theta|\text{H}\rangle + \sin\theta|\text{V}\rangle. \tag{1.26}$$

A polarizer is a measuring device. It has two eigenvalues 1 and 0. The eigenstate corresponding to the eigenvalue 1 is the state of linear polarization along the transmission axis, and eigenstate corresponding to the eigenvalue 0 is the state of linear polarization perpendicular to the transmission axis of the polarizer. Moreover, the polarizer has also a functional character. That is, the photon that is transmitted is the one with eigenvalue 1. The photon that is blocked by the polarizer has the eigenvalue 0. If a polarizer has its transmission axis along the horizontal direction, then the probability that a photon in state $|\theta\rangle$ gets transmitted is

$$\mathcal{P}_\theta = \cos^2\theta. \tag{1.27}$$

After the polarizer, the photon is in state $|H\rangle$.

Exercise 5 If the transmission axis of the polarizer is at an angle θ with the horizontal, what is the probability that an incident photon in state $|H\rangle$ gets transmitted? What is the state of the photon after the polarizer?

Chapter 2

Operators

Any measurement of an observable A is represented by an operator $\hat{A}$ that acts on the state kets. Operators obey the eigenvalue equation. In this equation an operator acting on one of its eigenstates produces the eigenvalue multiplying the eigenstate:

$$\hat{A}|a_i\rangle = a_i|a_i\rangle. \tag{2.1}$$

In the case of the spin of a photon the spin operator is represented by $\hat{S}$, so applying its definition

$$\hat{S}|R\rangle = -\hbar|R\rangle \tag{2.2}$$
$$\hat{S}|L\rangle = +\hbar|L\rangle. \tag{2.3}$$

An operator not acting on one of its eigenstates would then transform the state into another. Next we look at special cases of operators. For example, the NOT operator $\hat{X}$ is one that lives in a 2-dimensional space with state vectors $|0\rangle$ and $|1\rangle$, and which transforms one state into another:

$$\hat{X}|0\rangle = |1\rangle \tag{2.4}$$
$$\hat{X}|1\rangle = |0\rangle \tag{2.5}$$

The not operator is similar to the 90-degree mirror in optical systems. If a photon can travel in the x and y directions, we can define the states by their direction: $|x\rangle$ and $|y\rangle$. The 90-degree mirror operator $\hat{M}$ transforms one into the other, and thus is a NOT operator:

$$\hat{M}|x\rangle = |y\rangle \tag{2.6}$$
$$\hat{M}|y\rangle = |x\rangle \tag{2.7}$$

The beam-splitter has a reflection coefficient r and a transmission coefficient t. When the beam splitter, with operator $\hat{B}$ acts on a photon coming along the x-direction, its splits its amplitude:

$$\hat{B}|x\rangle = t|x\rangle + r|y\rangle, \tag{2.8}$$

where $r = i/\sqrt{2}$ and $t = 1/\sqrt{2}$.

Exercise 6 An interferometer has two beam splitters and two mirrors, as shown in Fig. 2.1. A photon reaching the interferometer first encounters a beam splitter. Its amplitudes then encounter a mirror, and finally they encounter the second beam splitter. If the state of an incoming photon is $|x\rangle$, the interferometer operation is equivalent to consecutive action of the operators $\hat{B}$, $\hat{M}$ and $\hat{B}$. Find the probability of the photon ending in state $|x\rangle$ after the interferometer.

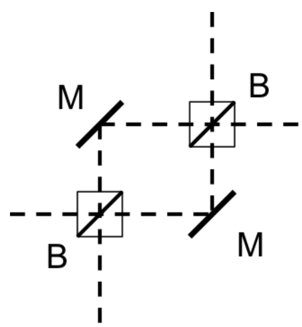

Figure 2.1: A Mach-Zehnder interferometer.

2.1 Identity Operator

The identity operator $\hat{I}$ is an operator that acting on a vector produces the same vector. That is, it does nothing:

$$\hat{I}|\psi\rangle = |\psi\rangle. \tag{2.9}$$

The NOT operator applied twice is the identity operator: $\hat{X}\hat{X} = \hat{I}$. This can be also understood with the optics example of 2 mirrors acting like a periscope, as shown in Fig. 2.2.

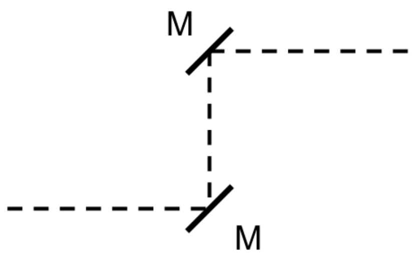

Figure 2.2: A mirror operating twice in a periscope is a unitary operator.

2.2 Adjoint Operator

An operator acting on a ket, can be represented by

$$\hat{A}|\psi\rangle = c|\phi\rangle, \tag{2.10}$$

The complex conjugate of the state on the right is

$$(c|\phi\rangle)^* = c^*\langle\phi| = \langle\psi|\hat{A}^\dagger \tag{2.11}$$

The operator $\hat{A}^\dagger$ is the adjoint of $\hat{A}$.

2.3 Unitary Operator

An operator $\hat{U}$ satisfying

$$\hat{U}^\dagger\hat{U} = \hat{I} \tag{2.12}$$

is called unitary. Unitary operators preserve the norm of the state.

Exercise 7 Show that the NOT operator is unitary.

2.4 Projection Operator

The projection operator projects a state onto the eigenstate of the operator. The projection operator with eigenstate $|\psi\rangle$ is given by

$$\hat{P}_\psi = |\psi\rangle\langle\psi|. \tag{2.13}$$

A polarizer can be represented by a projection operator. A polarizer with transmission axis along the horizontal direction acting on the state $|\theta\rangle$ of Eq. 1.26 is given by

$$\hat{P}_H|\theta\rangle = |H\rangle\langle H|\theta\rangle = |H\rangle\cos\theta. \tag{2.14}$$

Exercise 8 An incident photon is in state $|\theta\rangle$. It encounters a polarizer along the horizontal direction. Then past this polarizer is another polarizer with transmission axis along the vertical direction. Show using projection operators that the probability of transmission after the second polarizer is 0.

Exercise 9 Now in the previous example we insert a polarizer aligned along the diagonal state in between the two polarizers, as shown in Fig. 2.3. The diagonal state is given by:

$$|D\rangle = \frac{1}{\sqrt{2}}|H\rangle + \frac{1}{\sqrt{2}}|V\rangle \tag{2.15}$$

Apply the projection operators consecutively to find the probability that the photon gets transmitted after the last polarizer.

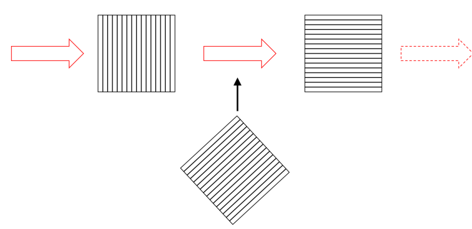

Figure 2.3: Two polarizers are orthogonal to each other. A third one rotated by 45 degrees is inserted in between.

Let us consider a space spanned by eigenstates $|a_i\rangle$. A general state can be expressed as a superposition of eigenstates:

$$|\psi\rangle = \sum_i c_i|a_i\rangle, \tag{2.16}$$

but we know that

$$c_i = \langle a_i|\psi\rangle \tag{2.17}$$

Thus, replacing into Eq. 2.16 we get

$$|\psi\rangle = \sum_i \langle a_i|\psi\rangle |a_i\rangle, \tag{2.18}$$

$$= \sum_i |a_i\rangle\langle a_i|\psi\rangle \tag{2.19}$$

$$= \left(\sum_i |a_i\rangle\langle a_i|\right)|\psi\rangle \tag{2.20}$$

So we are left to conclude that

$$\sum_i \hat{P}_i = \sum_i |a_i\rangle\langle a_i| = 1. \tag{2.21}$$

We can apply the act of making a measurement by applying the projection operator. Suppose that we have a photon in the diagonal state of Eq. 2.15. We place a polarizer with transmission axis horizontal in the path of the photon. We represent this operation by

$$\hat{P}_H|D\rangle = |H\rangle\langle H|D\rangle \tag{2.22}$$

$$= \frac{1}{\sqrt{2}}|H\rangle. \tag{2.23}$$

The probability of finding the photon in state $|H\rangle$ is

$$\mathcal{P}_H = \left|\hat{P}_H|D\rangle\right|^2 = \langle D|H\rangle|H\rangle\langle H||H\rangle D\rangle = \frac{1}{2} \tag{2.24}$$

Note that the last equation implied a property of the projection operator:

$$\hat{P}\hat{P} = \hat{P} \tag{2.25}$$

Furthermore, the state of the light is not the one given by Eq. 2.23 because it is not normalized. The state of the system initially in state $|\phi\rangle$ after a measurement that projects it into a state $|\psi\rangle$ is given by

$$|\psi\rangle = \frac{\hat{P}_\psi|\phi\rangle}{\sqrt{\langle\phi|\hat{P}_\psi|\phi\rangle}} \tag{2.26}$$

This equation seems silly and circular, but it will become much more useful when we apply it to a situation with more than one degree of freedom.

2.5 Hermitian Operator

An operator $\hat{A}$ is Hermitian if satisfies

$$\hat{A}^\dagger = \hat{A}. \tag{2.27}$$

That is, it is equal to its adjoint. It is also called a self-adjoint operator. These operators are special because their eigenvalues are real.

Exercise 10 Show that the eigenvalues a_i of a self adjoint operator $\hat{A}$ are real. You can do this by calculating the inner product of $\hat{A}|a_i\rangle$ with its complex conjugate and apply Eq. 2.11.

Chapter 3

Photon Games

We now apply the current formalism to a photon living in a 2-dimensional space. We will use two types of spaces: one where the degree of freedom is the momentum space (i.e., direction of propagation along the x and y axes), and another one where the degree of freedom is the polarization. We will also introduce the matrix notation to simplify calculations.

3.1 Momentum Space

The matrix representation of the state vectors is

$$|x\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.1}$$

and

$$|y\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{3.2}$$

The beam splitters mentioned earlier either reflect or transmit a photon. The operators for a symmetric beam-splitter applied to the state vectors are

$$\hat{B}|x\rangle = r|y\rangle + t|x\rangle \tag{3.3}$$

$$\hat{B}|y\rangle = t|y\rangle + r|x\rangle. \tag{3.4}$$

The matrix representation of the beam splitter is

$$\hat{B} = \begin{pmatrix} t & r \\ r & t \end{pmatrix}, \tag{3.5}$$

It is straightforward to verify Eqs. 3.3 and 3.4 using this matrix notation. Suppose we have a Mach-Zehnder interferometer as shown in Fig. 3.1, it has 2 beam splitters and 2 mirrors. In a real interferometer the arms do not have the same length, so we need to account the phase that the amplitude accrues in each arm. We do this with the phase shift operator

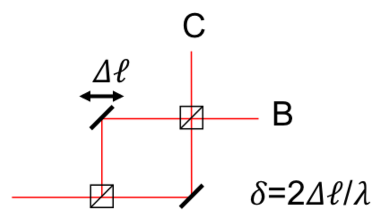

Figure 3.1: A Mach-Zehnder interferometer for single photons.

$$\hat{G} = \begin{pmatrix} e^{i\delta_1} & 0 \\ 0 & e^{i\delta_2} \end{pmatrix}, \tag{3.6}$$

where $\delta_i = 2\pi\ell_i/\lambda$ with ℓ_i being the length of distance traveled in arm i. The phase operator $\hat{G}$ is unitary, so the state vectors are eigenstates of the phase-propagating operator. The State of the light after the interferometer is

$$|\phi_0'\rangle = \hat{Z}|\phi_0\rangle \tag{3.7}$$

where

$$\hat{Z} = \hat{B}\hat{G}\hat{M}\hat{B}. \tag{3.8}$$

Note that all the operators representing these optical elements are unitary. The interferometer as a whole is also represented by a unitary operator.

Exercise 11 Find an expression for $\hat{Z}$

Suppose that

$$\hat{Z} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{3.9}$$

If the initial state is $|\phi_0\rangle = |x\rangle$, then the final state is:

$$|\phi_0'\rangle = a|x\rangle + c|y\rangle \tag{3.10}$$

Exercise 12 If we do an experiment with single photons and put detectors at the two outputs of the interferometer at B and C, what is the probability that the photon ends at B (in the state $|x\rangle$) after going through the interferometer? What is the probability that the photon ends at C (in the state $|y\rangle$)?

We start with the photon in an input state going along x. The first beam splitter puts the photon in a superposition of states (x and y). The second beam splitter puts the state of the light also into a superposition of output states. When we do a measurement the probability amplitude is not a constant number: the total probability amplitude for ending in $|x\rangle$ is the superposition of the probability amplitudes for going through both arms. The probability shows *interference*. Figure 3.2 shows the results of a real single-photon experiment. In the y-axis we get photon counts in the detectors, and in the x-axis of the graph we have the phase difference due to the two arms of the interferometer $\delta = \delta_1 - \delta_2$. This phase is changed by changing the difference in length between the two arms $\Delta\ell$, as shown in Fig. 3.1. We can see in the graph that the number of counts in B and C oscillates as the phase changes. Those graphs show interference, constructive where there is a maximum and destructive where there is a minimum.

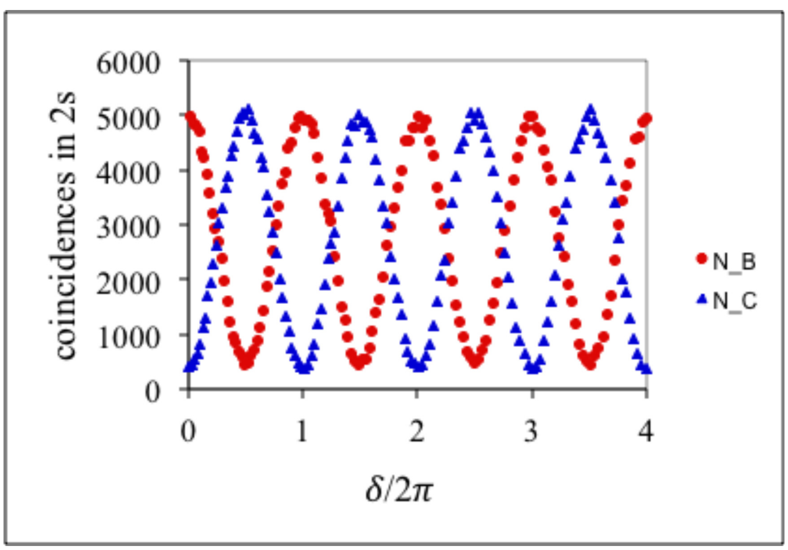

Figure 3.2: Counts recorded at detectors B and C at the output ports of a Mach-Zehnder interferometer.

The interesting part of this is that in quantum mechanics superposition means that the path that the photon takes is undefined. This notion already challenges *realism*, the idea that there is an element of reality to everything. In our case, the path of the photon. Quantum mechanics says that it is undefined! Moreover, every time we detect light, we detect whole photons, not half or fractional photons. So how does it go both ways? Quantum mechanics says nothing about it. That is up to us, and that is when we get into trouble. Paul Dirac had a famous statement: "...each photon interferes only with itself," which feels like a contradiction: something whole going two separate ways. That is not all. Detectors B and C in Fig. 3.1 detect photons. Quantum mechanics only states the probability that the photon is detected. In the graph of Fig. 3.2 we show detector recordings in 2 seconds. Any given photon will appear in detector B or C, but not in both. What about right before being detected? We do not know. A thought introduced by john Wheeler is that the photon extends both ways, but is detected in only one place. It "pops" in one of the detectors, and the detector "clicks." The detectors can in principle be at the two ends of the universe, and still only one clicks. All the counts in detectors in B and C of Fig. 3.2 are produced by different photons.

Note that this is not particular to photons. One could do superposition of electrons, atoms, etc.

3.2 Polarization Space

If we use polarization as a degree of freedom, and since it is a 2-dimensional space, we can define vectors the same way we did for momentum states. This space corresponds to a quantum bit or qubit. This way we define the states of polarization:

$$|H\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.11}$$

for the state of a horizontally polarized photon, and

$$|V\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.12}$$

for vertically polarized.

The states $|H\rangle$ and $|V\rangle$ are a useful basis, but there are others, such as the rotated bases. We get to them by rotating the $|H\rangle$ and $|V\rangle$ basis via the unitary rotation operator:

$$\hat{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{3.13}$$

representing a rotation by θ.

Example

If we rotate states $|H\rangle$ and $|V\rangle$ by $\theta = \pi/4$ we get

$$\begin{aligned} \hat{R}_{\pi/4}|H\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} && (3.14) \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} && (3.15) \\ &= \frac{1}{\sqrt{2}}\left(|H\rangle + |V\rangle\right) && (3.16) \end{aligned}$$

and

$$\begin{aligned} \hat{R}_{\pi/4}|V\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} && (3.17) \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix} && (3.18) \\ &= \frac{1}{\sqrt{2}}\left(-|H\rangle + |V\rangle\right). && (3.19) \end{aligned}$$

We define

$$\begin{aligned} |D\rangle &= \frac{1}{\sqrt{2}}\left(|H\rangle + |V\rangle\right) && (3.20) \\ |A\rangle &= \frac{1}{\sqrt{2}}\left(-|H\rangle + |V\rangle\right). && (3.21) \end{aligned}$$

Exercise 13 If the rotation operator is given by Eq. 3.29, find the state $|\theta\rangle$ representing $|H\rangle$ rotated by an angle θ.

3.2.1 Polarization Operators

In a previous section we represented a polarizer by a projection operator. The polarizer performs a non-unitary operation; it performs a projection. If the polarizer is aligned horizontally (i.e. with zero rotation), its matrix is given by:

$$\hat{P}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{3.22}$$

Exercise 14 Show that this is consistent with $\hat{P}_0 = |H\rangle\langle H|$.

The rotated polarizer should be equivalent to

$$\hat{P}_\theta = |\theta\rangle\langle\theta|. \tag{3.23}$$

An alternative way to get the rotated polarizer is

$$\hat{P}_\theta = \hat{R}\hat{P}_0\hat{R}^\dagger. \tag{3.24}$$

The matrix form of the adjoint of a matrix is the complex conjugate of the transpose matrix. Since $\hat{R}$ has real elements, its adjoint is just the transpose matrix

$$\hat{R}_\theta^\dagger = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{3.25}$$

We can realize that the adjoint of the rotation operator is the rotation operator rotating the opposite angle:

$$\hat{R}_\theta^\dagger = \hat{R}_{-\theta}. \tag{3.26}$$

We can also verify that the rotation operator is unitary.

In the optics bag of tricks we have the half-wave plate. It is a birefringent device that inserts a π phase between the linear components parallel and perpendicular to its optic axis. When its axis forms an angle of 0 degrees with the horizontal it is represented by the operator:

$$\hat{W}_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.27}$$

To appreciate what half-wave plate does we need to rotate it. This is done by rotating the operator an angle θ:

$$\hat{W}_\theta = \hat{R}\hat{W}_0\hat{R}^\dagger \tag{3.28}$$

Exercise 15 Show that the matrix for the rotated half-wave plate is given by:

$$\hat{W}_\theta = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} \tag{3.29}$$

It is easy to show that

$$\hat{W}_{\pi/4}|H\rangle = |V\rangle. \tag{3.30}$$

Other important polarization states are the circular states: right circular,

$$|R\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}, \tag{3.31}$$

and left circular

$$|L\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \tag{3.32}$$

These states can be created with a quarter-wave plate. This is another birefringent device, with matrix given by

$$\hat{Q} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \tag{3.33}$$

We can rotate the quarter-wave plate. If we do this by 45 degrees, we get a device that transforms vertical and horizontal polarization into right and left circular, and viceversa.

Exercise 16 Verify the previous statement.

3.2.2 Polarization Qubits

In the context of qubits we call $|0\rangle = |H\rangle$ and $|1\rangle = |V\rangle$, and so in the diagonal basis $|+\rangle = |D\rangle$ and $|-\rangle = |A\rangle$. Some of the most important gates for qubits: are implemented by a half-wave plate:

$$\hat{Z} = \hat{W}_0 \tag{3.34}$$

$$\hat{X} = \hat{W}_{\pi/4} \tag{3.35}$$

$$\hat{H} = \hat{W}_{\pi/8}. \tag{3.36}$$

Exercise 17 Show $\hat{X}|1\rangle = |0\rangle$.

Exercise 18 Show $\hat{Z}|+\rangle = |-\rangle$.

Exercise 19 Show $\hat{H}|-\rangle = |1\rangle$.

3.3 Two Degrees of Freedom: Momentum and Polarization

A momentum single photon in 2-dimensions corresponds to a qubit, and the state of polarization is another qubit. Thus we have 2 qubits. First we are going to set this experiment with bras and kets and then we will set it up with matrices.

3.3.1 Dirac Notation

Quantum mechanics has a method to generate vectors and operator matrices of combined Hilbert spaces: it involves the tensor product, which is denoted by the symbol $\otimes$. It stands to separate the effects of the two spaces separately. This way, a photon traveling along the x-direction vertically polarized is given by

$$|x\rangle \otimes |V\rangle \tag{3.37}$$

In this case we have four eigenstates. The other three are: $|x\rangle \otimes |H\rangle$, $|y\rangle \otimes |H\rangle$, $|y\rangle \otimes |V\rangle$. Let us send the photon through a beam splitter. A non-polarizing beam splitter does not affect the polarization, so the operation for this case is

$$\begin{aligned}
(\hat{B} \otimes \hat{I})|x\rangle \otimes |V\rangle &= \hat{B}|x\rangle \otimes \hat{I}|V\rangle & (3.38)\\
&= (t|x\rangle + r|y\rangle) \otimes |V\rangle, & (3.39)\\
&= t|x\rangle \otimes |V\rangle + r|y\rangle \otimes |V\rangle, & (3.40)\\
&= t|x\rangle|V\rangle + r|y\rangle|V\rangle, & (3.41)
\end{aligned}$$

where in the last step we have omitted the tensor-product symbol for simplicity. The use of operator $\hat{I}$ means that the beam splitter does nothing in the subspace of polarization. It is important to keep the order of the operators and the states. Equation 3.41 represents the superposition of a photon traveling in the x and y directions, which is vertically polarized.

Let us analyze the experiment of the quantum eraser, shown in Fig. 3.3. After the mirrors (with one capable of moving to change the phase) there are two half wave plates. Then after the interferometer in the x-direction there is the possibility of adding a polarizer with transmission axis at 45-degrees with the horizontal.

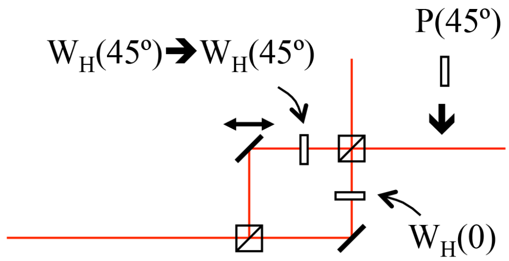

Figure 3.3: The quantum eraser apparatus.

The quantum eraser has three stages: indistinguishable paths, distinguishable paths and erased-path information. Let us go through each of these steps in detail:

1. *Indistinguishable Paths* In this stage both wave plates have their optic axis aligned with the horizontal, set to 0. The paths are indistinguishable, so we follow the state of the light the following way:

$$(\hat{B} \otimes \hat{I})(\hat{G} \otimes \hat{I})(\hat{W}_{H,0})(\hat{M} \otimes \hat{I})(\hat{B} \otimes \hat{I})|x\rangle \otimes |V\rangle \tag{3.42}$$

Step by step, after the first beam splitter:

$$\begin{aligned} |\psi^i\rangle &= (\hat{B} \otimes \hat{I})|x\rangle \otimes |V\rangle, & (3.43)\\ &= \hat{B}|x\rangle \otimes \hat{I}|V\rangle, & (3.44)\\ &= t|x\rangle|V\rangle + r|y\rangle|V\rangle, & (3.45) \end{aligned}$$

After the mirrors

$$\begin{aligned} |\psi^{ii}\rangle &= (\hat{M} \otimes \hat{I})|\psi^i\rangle & (3.46)\\ &= t\hat{M}|x\rangle \otimes \hat{I}|V\rangle + r\hat{M}|y\rangle \otimes \hat{I}|V\rangle, & (3.47)\\ &= t|y\rangle|V\rangle + r|x\rangle|V\rangle, & (3.48) \end{aligned}$$

Note that we have defined a dual space operator representing the waveplates in the two arms. In the first setting they do not change the state:

$$\begin{aligned} \hat{W}_{H,0}|x\rangle|V\rangle &= |x\rangle|V\rangle & (3.49)\\ \hat{W}_{H,0}|y\rangle|V\rangle &= |y\rangle|V\rangle & (3.50) \end{aligned}$$

So the wave plate operator in this case will do nothing:

$$|\psi^{iii}\rangle = \hat{W}_{H,0}|\psi^{ii}\rangle = |\psi^{ii}\rangle \tag{3.51}$$

Proceeding now with the phase operator

$$\begin{aligned} |\psi^{iv}\rangle &= (\hat{G}\otimes\hat{I})|\psi^{iii}\rangle & (3.52)\\ &= t\hat{G}|y\rangle\otimes\hat{I}|V\rangle + r\hat{G}|x\rangle\otimes\hat{I}|V\rangle, & (3.53)\\ &= te^{i\delta_2}|y\rangle|V\rangle + re^{i\delta_1}|x\rangle|V\rangle, & (3.54) \end{aligned}$$

Finally, applying the second beam splitter gives

$$\begin{aligned} |\psi^{v}\rangle &= (\hat{B}\otimes\hat{I})|\psi^{iv}\rangle & (3.55)\\ &= te^{i\delta_2}\hat{B}|y\rangle\otimes\hat{I}|V\rangle + re^{i\delta_1}\hat{B}|x\rangle\otimes\hat{I}|V\rangle, & (3.56)\\ &= (tte^{i\delta_2} + rre^{i\delta_1})|y\rangle|V\rangle + rt(e^{i\delta_1}+e^{i\delta_2})|x\rangle|V\rangle, & (3.57) \end{aligned}$$

This last equation is the final state of the light. The probability that the light comes out in the x direction is then

$$\begin{aligned} \mathcal{P}_x &= \left|rt(e^{i\delta_1}+e^{i\delta_2})|x\rangle|V\rangle\right|^2 & (3.58)\\ &= \left|rt(e^{i\delta_1}+e^{i\delta_2})\right|^2 & (3.59)\\ &= \frac{1}{2}(1+\cos\delta), & (3.60) \end{aligned}$$

where $\delta = \delta_1 - \delta_2$. This implies that the probability shows interference. An experiment with single photons shows interference. The diamond symbols of Fig. 3.4 represent data as a function of the phase of the interferometer shown in Fig. 3.3. The data shows interference, as discussed previously, and corresponds to the case discussed here where the paths are indistinguishable.

2. *Distinguishable Paths.* In this stage we rotate one of the wave plates by 45 degrees so that it rotates the polarization of the light going through it. Thus the wave plate operator changes to $\hat{W}_{H,45/0}$, which acts on the state in the following way:

$$\begin{aligned} \hat{W}_{H,45}|x\rangle\otimes|V\rangle &= |x\rangle|H\rangle & (3.61)\\ \hat{W}_{H,45}|y\rangle\otimes|V\rangle &= |y\rangle|V\rangle & (3.62) \end{aligned}$$

This changes $|\psi^{iii}\rangle$ above to

$$\begin{aligned} |\psi^{iii}\rangle &= \hat{W}_{H,45}|\psi^{ii}\rangle & (3.63)\\ &= t\hat{W}_{H,45}|y\rangle|V\rangle + r\hat{W}_{H,45}|x\rangle|V\rangle, & (3.64)\\ &= t|y\rangle|V\rangle + r|x\rangle|H\rangle, & (3.65)\end{aligned}$$

This change affects the subsequent states after phase and second beam splitter operators:

$$|\psi^{iv}\rangle = te^{i\delta_2}|y\rangle|V\rangle + re^{i\delta_1}|x\rangle|H\rangle, \quad (3.66)$$

$$|\psi^{v}\rangle = tte^{i\delta_2}|y\rangle|V\rangle + rre^{i\delta_1}|y\rangle|H\rangle + rte^{i\delta_1}|x\rangle|H\rangle + tre^{i\delta_2}|x\rangle|V\rangle \quad (3.67)$$

Note that all four terms contain states that are mutually orthogonal. So, if we calculate the probability that the photon leaves the interferometer in the x direction we get

$$\begin{aligned} \mathcal{P}_x &= \left|rte^{i\delta_1}|x\rangle|H\rangle + tre^{i\delta_2}|x\rangle|V\rangle\right|^2 & (3.68)\\ &= \left|rte^{i\delta_1}\right|^2 + \left|tre^{i\delta_2}\right|^2 & (3.69)\\ &= \frac{1}{2}. & (3.70)\end{aligned}$$

That is, in this situation there is no interference because the probability is constant. The square symbols in the data of Fig. 3.4 corresponds to this situation. As can be seen, as the phase difference is changed the data shows no evidence of interference, in accordance with the predictions of quantum mechanics. We can interpret this result by saying that there is path information encoded in the polarization, and so there is no interference. Note that we did not measure that path information. Interference disappears even if in principle we can obtain the path information.

3. *Erasing the Path Information* The last step in the eraser is to erase the path information. This is done in a peculiar way: by adding a polarizer *after* the interferometer. It sounds counterintuitive that after the interferometer we decide whether the photon should interfere with itself or not. Remember that the action of a polarizer with transmission axis at 45 degrees to the horizontal is to project the state along the eigenstate of the polarizer, which in this case is given by

$$\hat{P}_{45} = |D\rangle\langle D|. \quad (3.71)$$

But applying it to orthogonal states gives:

$$\hat{P}_{45}|H\rangle = |D\rangle\langle D|H\rangle = \frac{1}{\sqrt{2}}|D\rangle \quad (3.72)$$

$$\hat{P}_{45}|V\rangle = |D\rangle\langle D|V\rangle = \frac{1}{\sqrt{2}}|D\rangle. \quad (3.73)$$

By inspecting the previous two equations, or more rigorously by applying Eq. 2.26, lead us that the photon traveling through any of the two paths is in the same state after the polarizer. At this point we now want to calculate the probability of detecting the light traveling in the x direction after the polarizer. We could do this analytically by first applying two projection operators, one to select the x direction, and the other, the polarizer.

$$\mathcal{P}_{x,D} = |(|x\rangle\langle x| \otimes |D\rangle\langle D|)|\psi^v\rangle|^2 \quad (3.74)$$

$$\mathcal{P}_{x,D} = \left| rte^{i\delta_1}\frac{1}{\sqrt{2}}|x\rangle|D\rangle + tre^{i\delta_2}\frac{1}{\sqrt{2}}|x\rangle|D\rangle \right|^2 \quad (3.75)$$

$$= \left| rt\frac{1}{\sqrt{2}}(e^{i\delta_1} + e^{i\delta_2}) \right|^2 \quad (3.76)$$

$$= \frac{1}{4}(1 + \cos\delta). \quad (3.77)$$

That is, the interference reappears, or put differently, the polarizer erased the path-distinguishing information. Note that the circular symbols in the data of Fig. 3.4 show interference, and the maximum counts go half way, as predicted.

3.3.2 Matrix Notation

Let us redo the previous analysis with the matrix notation. In the tensor product operation, we multiply each element of one space (propagation direction) to each element of the other space (polarization). The ordering of spaces in the tensor product is important. In our case, we will order direction of propagation first, and polarization second.

For example, if we have a vector $|A\rangle$ in the space of propagation directions, and a vector $|B\rangle$ in the space of polarization, the tensor product of two vectors

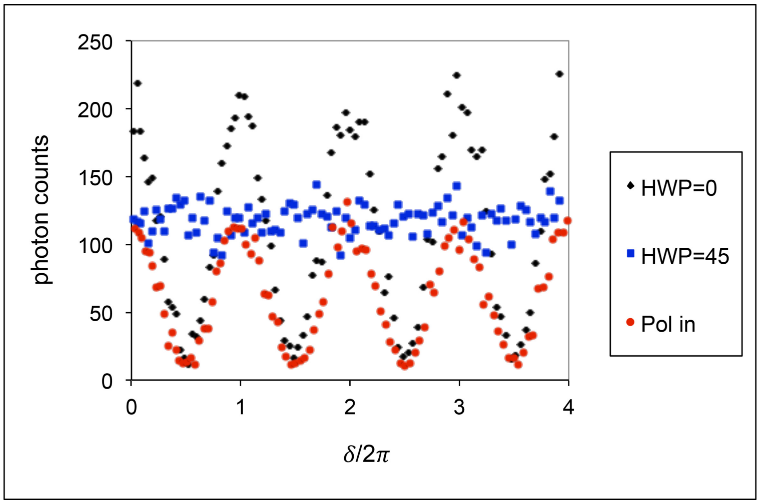

Figure 3.4: Data on the quantum eraser. Diamonds are the indistinguishable case; squares correspond to the distinguishable case, and circles correspond to the eraser.

is:

$$|A\rangle|B\rangle = |A\rangle \otimes |B\rangle = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \otimes \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \tag{3.78}$$

$$= \begin{pmatrix} a_1 \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \\ a_2 \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} a_1 b_1 \\ a_1 b_2 \\ a_2 b_1 \\ a_2 b_2 \end{pmatrix}. \tag{3.79}$$

The eigenstates of our experiment are then:

$$|x\rangle|H\rangle = |x\rangle \otimes |H\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{3.80}$$

$$|x\rangle|V\rangle = |x\rangle \otimes |V\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}. \tag{3.81}$$

Exercise 20 Find the vectors for $|y\rangle|H\rangle$ and $|y\rangle|V\rangle$.

The matrices for the operators in the larger space are the tensor product of the matrices of the operators that act on each space. For example, an operator in the direction of propagation space, $\hat{A}$, combines with an operator in the polarization space, $\hat{B}$, the following way:

$$\hat{A} \otimes \hat{B} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \otimes \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \tag{3.82}$$

$$= \begin{pmatrix} a_1 \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} & a_2 \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \\ a_3 \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} & a_4 \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \end{pmatrix} \tag{3.83}$$

$$= \begin{pmatrix} a_1 b_1 & a_1 b_2 & a_2 b_1 & a_2 b_2 \\ a_1 b_3 & a_1 b_4 & a_2 b_3 & a_2 b_4 \\ a_3 b_1 & a_3 b_2 & a_4 b_1 & a_4 b_2 \\ a_3 b_3 & a_3 b_4 & a_4 b_3 & a_4 b_4 \end{pmatrix} \tag{3.84}$$

Notice that the ordering procedure for the elements of the matrix is the same as for elements of the vectors.

Using the tensor product we can also construct the matrices for the interferometer. The beam splitter acts on one space and not the other, so it will be the tensor product of the beam-splitter matrix (first) with the identity (second). We put identity for the polarization part because the beam splitter does not alter the polarization. The matrix for the beam splitter in the larger space will be:

$$\hat{B} \otimes \hat{I} = \begin{pmatrix} t & r \\ r & t \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} t & 0 & r & 0 \\ 0 & t & 0 & r \\ r & 0 & t & 0 \\ 0 & r & 0 & t \end{pmatrix}, \tag{3.85}$$

where $t = 1/\sqrt{2}$ and $r = i/\sqrt{2}$.

Exercise 21 Verify that the beam-splitter operator $\hat{B} \otimes \hat{I}$ acting $|x\rangle|V\rangle$ does not alter the polarization of the state.

Exercise 22 Find the matrix for the mirrors of the interferometer: $\hat{M} \otimes \hat{I}$.

Exercise 23 Find the matrix for the interferometer phase: $\hat{G} \otimes \hat{I}$, where δ is the phase difference between the two arms, and:

$$\hat{G} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\delta} \end{pmatrix}$$

.

The quantum eraser has two wave plates in the arms of the interferometer: a half wave plate in the upper arm with angle θ and a half-wave plate with $\theta = 0$ in the lower arm. This is represented analytically in the following way:

$$\hat{W}_{\theta,0} = |x\rangle\langle x| \otimes \hat{W}_{H,\theta} + |y\rangle\langle y| \otimes \hat{W}_{H,0}. \tag{3.86}$$

The matrix representing it is:

$$\hat{W}_{\theta,0} = \begin{pmatrix} \cos 2\theta & \sin 2\theta & 0 & 0 \\ \sin 2\theta & -\cos 2\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{3.87}$$

Exercise 24 Find an expression for the full interferometer matrix (laborious): $\hat{Z} = (\hat{B} \otimes \hat{I})(\hat{W}_{H,\theta/0})(\hat{A} \otimes \hat{I})(\hat{M} \otimes \hat{I})(\hat{B} \otimes \hat{I})$.

Exercise 25 Verify that when $\theta = \pi/4$

$$\hat{Z} = \frac{1}{2} \begin{pmatrix} ie^{i\delta} & i & ie^{i\delta} & 1 \\ i & -ie^{i\delta} & 1 & e^{i\delta} \\ ie^{i\delta} & -1 & ie^{i\delta} & i \\ i & -e^{i\delta} & i & -ie^{i\delta} \end{pmatrix}, \tag{3.88}$$

Exercise 26 Indistinguishable paths: Calculate the probability of photons in state $|x\rangle|V\rangle$, and entering the interferometer with the waveplate at $\theta = 0$, exit the interferometer in the same state.

Exercise 27 Find the final state of the light when the wave plate is rotated an angle θ.

Exercise 28 Find the probability of the photon leaving the interferometer in the x-direction and with polarization vertical, as a function of θ.

Exercise 29 Fully Distinguishable Paths: Find the probability that the photon leaving the interferometer in the x-direction and with polarization vertical, for $\theta = \pi/4$.

Exercise 30 The Eraser: Find the probability that the photon leaving the interferometer with $\theta = \pi/4$ in the x- direction, and with polarization diagonal after being projected by a polarizer at 45 degrees.

Another interesting device is the polarization beam splitter, which transmits horizontally polarized light but reflects vertically polarized light. We can express it as

$$\hat{B}_p = |I\rangle \otimes |H\rangle\langle H| + \hat{M} \otimes |V\rangle\langle V|. \tag{3.89}$$

Exercise 31 Find the matrix expression for the polarization beam splitter.

3.4 Two Degrees of Freedom: Polarization and Spatial Mode

3.4.1 Spatial Mode: Photon's Image

Photons emitted by a source carry a spatial probability distribution that is known as the spatial mode. A common example of this is the transverse mode of a laser. It normally is in the fundamental Gaussian mode, but by making modifications to the laser cavity one can have the light exiting the laser to be in a high-order spatial mode. These spatial modes are solutions of the wave equation, so that the mode or shape that the light has remains constant as the light propagates. The spatial modes come in families of solutions to the wave equation, that depend on the chosen reference frame. For example, Hermite-Gauss modes are solutions to the wave equation in rectangular coordinates. Similarly, Laguerre-Gauss modes are solutions to the wave equation in cylindrical coordinates. The eigenmodes in one family can be expressed as linear combination of the modes in the other family. Spatial modes can also be generated by transmission through custom optical elements or by diffraction.

The whole concept of spatial mode is unique in that it reveals a that photons are not tiny particles. Photons have a wave aspect associated to them, and so they must be solutions to a wave equation and carry a spatial mode. In the photon sense one can think of the spatial mode as a transverse probability amplitude. Richard Feynman called photons "waveicles." When one sees the image of photons impinging on a camera, the first reaction is that the photons are the dots. No! Those are the pixels of the camera that can only detect whole photons. Figure 3.5(a) shows the image of the spatial mode of a single photon accumulated over many single-photon detections by an imaging system. The light is in a Laguerre-Gauss mode of order 1, which has a distinctive doughnut shape.

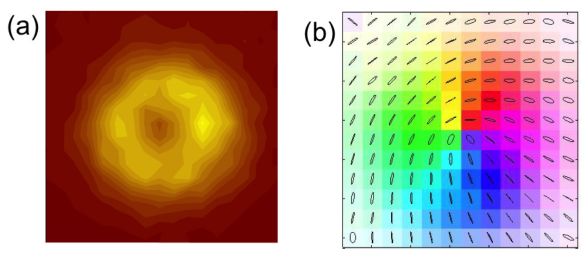

Figure 3.5: Images of (a) the spatial mode of a single photon, and (b) the vector mode of a single photon. The color in (b) corresponds to the orientation of the polarization of the mode. Ellipses mark the measured polarization at that location.

3.4.2 Vector Modes

Vector modes are non-separable superpositions of spatial mode and polarization. For example, a vector state could be of the form

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|u_A\rangle|H\rangle + |u_B\rangle|V\rangle\right), \tag{3.90}$$

where u_A and u_B represent orthogonal spatial modes. Photons prepared in this state are similar to the ones we discussed earlier in terms of momentum and polarization. However, in this case a photon traveling through space in a given direction carries this multi-dimensional information. It is a state that has been referred to as self-entangled or classically entangled. Both denominations are controversial, as many researchers object to such a classification.

The interesting aspect of the vector state is that it does not have a well defined polarization. In, fact when one measures the polarization via polarimetry one finds that the state of polarization varies from point to point. Laguerre-Gauss modes are characterized by a parameter called the topological charge. The vector mode of Fig. 3.5(b) is the measurement of the vector mode of the light when u_A and u_B are modes with topological charges 1 and 0, and the polarization states are right and left circular.

3.5 Two Photons in Polarization Space

Another way to have two qubits is to have two photons with each having a degree of freedom. The first case we will cover is when the degree of freedom is polarization.

3.5.1 Dirac Notation

If we denote the polarization states of a photon, horizontal and vertical, as $|H\rangle$ and $|V\rangle$, respectively, then the photon pairs that we created in previous labs would be in the state

$$|\psi\rangle = |V\rangle_1 \otimes |V\rangle_2 = |V\rangle_1 |V\rangle_2 \tag{3.91}$$

This state is called a "product state" because the wavefunction of the pair is the product of the wavefunctions of the two particles. The other three eigenstates are $|H\rangle_1|H\rangle_2$, $|H\rangle_1|V\rangle_2$ and $|V\rangle_1|H\rangle_2$.

If we decide to measure the two photons with polarizers set to angles θ_1 and θ_2 then we can use the projection operator for each photon

$$P_{\theta_1} \otimes P_{\theta_2}|\psi\rangle = |\theta_1\rangle\langle\theta_1|V\rangle_1|\theta_2\rangle\langle\theta_2|V\rangle_2 \tag{3.92}$$

$$= \sin\theta_1 \sin\theta_2 |\theta_1\rangle|\theta_2\rangle \tag{3.93}$$

The probability is then

$$\mathcal{P} = \sin^2\theta_1 \sin^2\theta_2 \tag{3.94}$$

New laboratory techniques allow the production of non-separable or entangled states. In particular, there are four important states also known as

Bell states:

$$|\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}}\left(|H\rangle_1|H\rangle_2 \pm |V\rangle_1|V\rangle_2\right), \tag{3.95}$$

$$|\Psi^{\pm}\rangle = \frac{1}{\sqrt{2}}\left(|H\rangle_1|V\rangle_2 \pm |V\rangle_1|H\rangle_2\right), \tag{3.96}$$

The importance of this state is that the states of photon 1 and photon 2 cannot be factored out. So measuring the state of one photon implied defining the state of the other photon. Suppose we have the pair in the state $|\Phi^+\rangle$. Measuring the state of photon 1 to be horizontal can be accomplished by putting a polarizer with transmission axis horizontal in the path of photon 1. We formalize this operation by projecting the state. The final state, after applying Eq. 2.26, is

$$|\psi\rangle = |H\rangle_1|H_2\rangle. \tag{3.97}$$

We note that before the measurement, the state of polarization the light is undefined, but after measuring the state of one of the photons, the state of the other is instantly defined. In this case to $|H\rangle_2$. Incredulous, Einstein derided it as "spooky action at a distance." This correlation is the basis for *nonlocality*; that the detection of one photon immediately "collapses" the wavefunction of the two, instantaneously at faster than the speed of light. This is the view advocated by Bohr in the so called "Copenhagen interpretation" of quantum mechanics. We stress though, that the state is $|\psi\rangle$, the joint state. We also note that this does not violate relativity because when we perform the measurement of photon 1, we do not know the outcome. Ahead of time we only know the probability of the outcomeL 1/2.

Let us study in more detail the entangled state $|\Phi^+\rangle$. What would the form of the state be in the diagonal-antidiagonal basis? If you recall, the diagonal basis states are related to the horizontal-vertical states by Eqs. 3.20 and 3.21.

Exercise 32 Put $|H\rangle$ and $|V\rangle$ in terms of $|D\rangle$ and $|A\rangle$ for each particle in state $|\Phi^+\rangle$ from Eq. 3.95, and show that

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\left(|D\rangle_1|D\rangle_2 + |A\rangle_1|A\rangle_2\right). \tag{3.98}$$

In the H-V basis the photons are in a superposition of being parallel to each other in two different ways. In the rotated basis they are also in an entangled state that is a superposition of the two possibilities in which they can be parallel! This is an interesting but unique aspect of state $|\Phi^+\rangle$.

Exercise 33 Show that in the diagonal basis state $|\Phi^-\rangle$ becomes

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}\left(|D\rangle_1|A\rangle_2 + |A\rangle_1|D\rangle_2\right). \tag{3.99}$$

That is, in state $|\Phi^-\rangle$ the light switches from being parallel in the H-V basis to being orthogonal in the D-A basis. Let us go back to $|\Phi^+\rangle$. Suppose that we now rotate the basis for each photon separately, by an angle θ_1 for photon 1 and θ_2 for photon 2. We already defined an expression for the vector $|\theta\rangle$ (Eq. 1.26). We need to define the state for the other vector $|\theta_\perp\rangle$, orthogonal to $|\theta\rangle$, so that together they can form a basis:

$$|\theta_\perp\rangle = -\sin\theta|H\rangle + \cos\theta|V\rangle \tag{3.100}$$

Then we replace the relations of Eq. 1.26 and 3.100 in the expression for $|\Psi^+\rangle$. If we do some algebra and group the terms we get

$$\begin{aligned}|\Phi^+\rangle =& \frac{1}{\sqrt{2}}[\cos(\theta_1-\theta_2)|\theta_1\rangle_1|\theta_2\rangle_2 + \cos(\theta_1-\theta_2)|\theta_{1\perp}\rangle_1|\theta_{2\perp}\rangle_2 \\ &+ \sin(\theta_1-\theta_2)|\theta_1\rangle_1|\theta_{2\perp}\rangle_2 + \sin(\theta_2-\theta_1)|\theta_{1\perp}\rangle_1|\theta_2\rangle_2]\end{aligned} \tag{3.101}$$

We now put polarizers project the state of each photon. Out of convenience, let us make the transmission axis of the polarizers be along the θ_1 and θ_2 directions. The probability of detecting a pair is

$$\mathcal{P}_{\theta_1,\theta_2} = |\langle\theta_1|_1\langle\theta_2|_2|\Phi^+\rangle|^2 = \frac{1}{2}\cos^2(\theta_1-\theta_2) \tag{3.102}$$

The previous equation encodes the correlations between the two photons. If $\theta_1 = \theta_2$ (parallel) the probability is maximum, and if they are $\theta_1 = \theta_2 \pm \pi/2$ (perpendicular) it is zero. If we express the pair in the same basis (i.e., $\theta_1 = \theta_2 = \theta$), we see that they are in the state

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\left(|\theta\rangle_1|\theta\rangle_2 + |\theta_\perp\rangle_1|\theta_\perp\rangle_2\right) \tag{3.103}$$

That is, they have a remarkable property: even though their polarization is undefined, it is in a superposition of both photons being linearly polarized parallel to each other, regardless of orientation. This type of correlation works for another Bell state: $|\Psi^-\rangle$, but not for the remaining two.

Exercise 34 Show that photons in state $|\Psi^-\rangle$ are in a superposition of always being orthogonal in *any* basis.

Yet, there are pagans. This the realistic view: photons' polarization is not undefined and in state $|\Phi^+\rangle$, they are just in $|H\rangle_1|H\rangle_2$ half the time, and in state $|V\rangle_1|V\rangle_2$ the remaining time. This situation is called the mixed state. How do we know that the photons are not in a mixed state? What is the probability of detection predicted for the mixed state? It is given by

$$\mathcal{P}_{\rm mix} = \frac{1}{2}\left|\langle\theta_1|_1\langle\theta_2|_2|H\rangle_1|H\rangle_2\right|^2 + \frac{1}{2}\left|\langle\theta_1|_1\langle\theta_2|_2|V\rangle_1|V\rangle_2\right|^2 \quad (3.104)$$

$$\mathcal{P}_{\rm mix} = \frac{1}{2}\cos^2\theta_1\cos^2\theta_2 + \frac{1}{2}\sin^2\theta_1\sin^2\theta_2. \quad (3.105)$$

Notice that Eqs. 3.102 and 3.105 have a different functional form. Thus, we have a chance to find out which one is correct. When $\theta_2 = 0$ both give the same answer:

$$\mathcal{P}_{\theta_1,\theta_2} = \mathcal{P}_{\rm mix} = (1/2)\cos^2\theta_1. \quad (3.106)$$

However, if $\theta_2 = \pi/4$ they give a different answer:

$$\mathcal{P}_{\theta_1,\pi/4} = (1/2)\cos^2(\theta_1 - \pi/4), \quad (3.107)$$

while

$$\mathcal{P}_{\rm mix} = 1/4. \quad (3.108)$$

In Fig. 3.6 we show the data obtained for the two cases discussed above: entangled and mixed. This distinction can already be considered as evidence of the distinction between entangled and mixed states. The measurements were done at about the same time. However, quantum mechanics does not specify a time or a place for the two measurements, So they could be done at distinct times and places and the results would be the same.

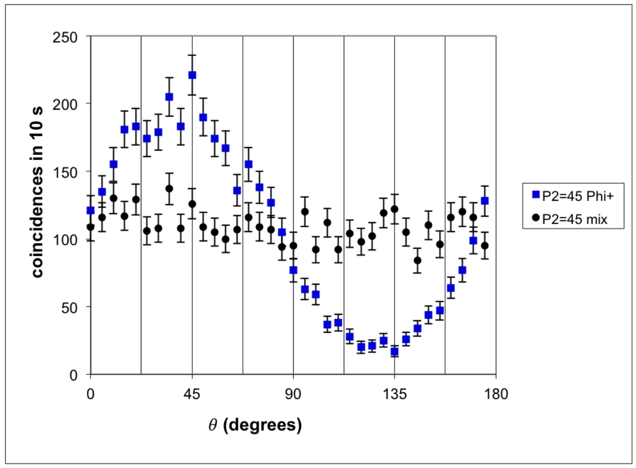

Figure 3.6: Data the measurement of coincident detections when the photons are in the entangled state $|\Phi^+\rangle$ (squares) and in the mixed state (circles) where the light is half the time in $|H\rangle_1|H\rangle_2$ and the other half in $|V\rangle_1|V\rangle_2$.

3.5.2 Matrix Form

Similar to the case of a single photon in two degrees of freedom we combine the vectors of each subspace using the tensor product. The eigenstates are

$$|H\rangle_1|H\rangle_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{3.109}$$

$$|H\rangle_1|V\rangle_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \tag{3.110}$$

$$|V\rangle_1|H\rangle_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \tag{3.111}$$

$$|V\rangle_1|V\rangle_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{3.112}$$

State $|\Phi^+\rangle$ is then given by

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \tag{3.113}$$

Operators will then follow the same tensor rules as before.

3.5.3 Production of Polarization-Entangled states

Spontaneous parametric down-conversion is a laboratory technique to produce pairs of photons from an input parent photon. Let us label the pump photon as photon "0," and the down-converted photons as "1" and "2." Energy is conserved in down-conversion, and so

$$E_0 = E_1 + E_2 \tag{3.114}$$

Momentum is also conserved. However, this occurs inside the crystal where the photon momentum is modified because $p = h/\lambda$ with $\lambda = \lambda_0/n$, where λ_0 is the wavelength in vacuum, and n is the index of refraction of the material. In our case, we will pick the down-converted photons when they have the same energy, or $E_1 = E_2$, and so they come out of the crystal at the same angle ($\alpha_1 = \alpha_2 = \alpha$). By tilting the crystal we can change α.

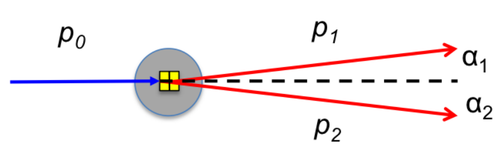

Figure 3.7: Momentum conservation in parametric down-conversion

Exercise 35 What is the wavelength of the down-converted photons (in vacuum)?

The polarization state of the down-converted light depends on the polarization of the pump beam and the type of crystal. With one Type-I crystal, if the pump beam is, say horizontally polarized, the downconverter photons

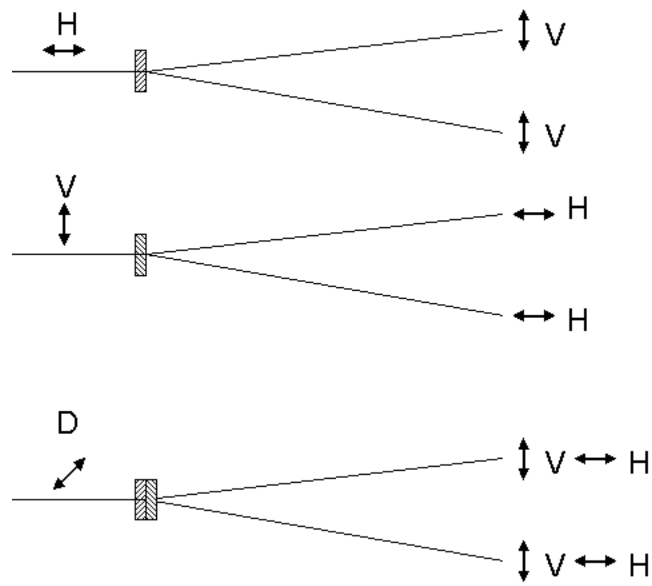

Figure 3.8: Method to produce polarization-entangled states: the bottom setup is a superposition of the two cases above.

are vertically polarized. If we change the polarization of the pump to vertical and not touch the crystal, we would not get down-conversion. However, if we rotate the crystal by 90° we get horizontally polarized pairs. A clever trick is to put two thin down-conversion crystals back to back but rotated by 90° with respect to each other. When we send a pump beam polarized at 45° to the pair of crystals, as shown in Fig. 3.8, the horizontal component of the pump polarization produces vertically polarized pairs with one crystal and the vertical component produces horizontally polarized pairs with the other crystal. If the crystal separation is thinner enough and if the crystal width is smaller than the beam width, then there is no way to tell in which crystal the photon pairs were created. Thus when the paths are indistinguishable the photon pairs get created into a state that is a superposition of the two possibilities:

$$|\Phi\rangle = \frac{1}{\sqrt{2}}\left(|H\rangle_1|H\rangle_2 + |V\rangle_1|V\rangle_2 e^{i\delta}\right), \tag{3.115}$$

where δ is a phase between the two possibilities, because they are not identical: in one case the pump creates the pairs in one crystal and the pairs travel at a certain speed through the second crystal; in the other case the pump travels through the first crystal at a different speed, and produces the pairs in the second crystal. In the lab we can put a birefringent crystal before the down-conversion to equilibrate the times that each polarization takes, as shown in Fig. 3.9, and so the phase is $\delta = 0$, and we get $|\Phi^+\rangle$. We note that we can get the other Bell states by putting wave plates in the path of the down-converted photons. For example, if we put a half wave plate with axis vertical in the path of photon 1 we introduce a phase of π in between $|H\rangle_1$ and $|V\rangle_2$. This phase gets introduced into the state $|\Phi^+\rangle$ to turn it into $|\Phi^-\rangle$.

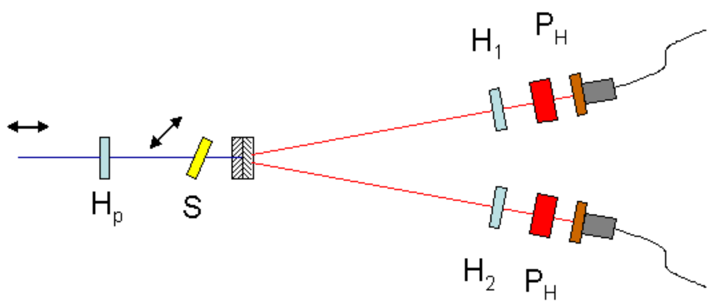

Figure 3.9: Apparatus to produce polarization-entangled photons and to characterize the state via a Bell test of by quantum state tomography. Optical elements are Half-wave plates (H), and polarizers (P), and phase-adjusting crystal (S).

Exercise 36 What optical elements do we have to add to make state $|\Psi^-\rangle$?

3.5.4 The Density Matrix

The Dirac notation in the vector form does not account for mixed states. A more formal way to treat mixed states within quantum mechanics is with the density matrix. Below we define the basic principles.

Pure State

For a pure state $|\psi\rangle$ the density matrix is defined as

$$\hat{\rho}_\psi = |\psi\rangle\langle\psi|. \tag{3.116}$$

That is, it is the outer product of the state vector with itself. As an example consider the state $|\Phi^+\rangle$. It is given in matrix form by

$$\hat{\rho}_{\Phi^+} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}. \tag{3.117}$$

Let us consider for a moment the density matrix of a product state, such as $|H\rangle_1|H\rangle_2$. It is the outer product of the vector of Eq. 3.109. It is not hard to see that the density matrix has element $\rho_{1,1} = 1/2$ (located in the upper-left corner), with $\rho_{i,j} = 0$ for other elements (i.e., zeros in the other locations). Similarly, the density matrix for state $|H\rangle_1|V\rangle_2$, will be a matrix with $\rho_{2,2} = 1/2$ (second along the diagonal) with zero otherwise. Similarly, a non-zero value for the other diagonal elements correspond to the cases of $|V\rangle_1|H\rangle_2$ and $|V\rangle_1|V\rangle_2$. There is a popular way to "graph" the density matrix. It consists of a 3-dimensional bar-graph, as shown in Fig. 3.12. The two dimensional plane locates the location of the element, with the third dimension, the bar, indicating its numerical value.

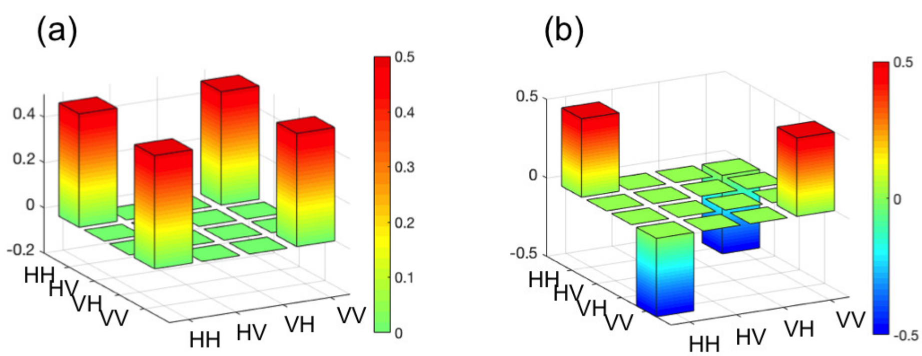

Figure 3.10: Graphical representation of the density matrices of two Bell states: $|\Phi^+\rangle$ (a) and $|\Phi^-\rangle$ (b).

Exercise 37 Find the density matrix for state $|\Psi^-\rangle$.

Some properties of the density matrix are:

- It has the form of the projector operator, so it satisfies

$$\hat{\rho}\hat{\rho} = \hat{\rho} \tag{3.118}$$

- Normalization is accomplished by the trace operation (the sum of the diagonal elements of the matrix):

$$\mathrm{Tr}(\hat{\rho}) = 1 \tag{3.119}$$

  This can be shown from the property of the trace. Suppose $\hat{\rho} = |\psi\rangle\langle\psi|$. If the eigenvectors of $\hat{\rho}$ are $|\phi_i\rangle$, then the trace of a matrix in that basis is

$$\mathrm{Tr}(\hat{A}) = \sum_i \langle\phi_i|\hat{A}|\phi_i\rangle. \tag{3.120}$$

  Applying this to the density matrix:

$$\begin{aligned}
\mathrm{Tr}(\hat{\rho}) &= \sum_i \langle\phi_i|\hat{\rho}|\phi_i\rangle && (3.121)\\
&= \sum_i \langle\phi_i|\psi\rangle\langle\psi|\phi_i\rangle && (3.122)\\
&= \sum_i \langle\psi|\phi_i\rangle\langle\phi_i|\psi\rangle && (3.123)\\
&= \langle\psi|\left(\sum_i |\phi_i\rangle\langle\phi_i|\right)|\psi\rangle && (3.124)\\
&= \langle\psi|\psi\rangle && (3.125)\\
&= 1. && (3.126)
\end{aligned}$$

- It is Hermitian:

$$\hat{\rho}^\dagger = \hat{\rho} \tag{3.127}$$

  As a consequence, its eigenvalues are real.

Other properties of the trace are:

$$\begin{aligned}
\mathrm{Tr}[\hat{A}\hat{B}] &= \mathrm{Tr}[\hat{B}\hat{A}], && (3.128)\\
\mathrm{Tr}[\hat{A}+\hat{B}] &= \mathrm{Tr}[\hat{A}] + \mathrm{Tr}[\hat{B}], && (3.129)\\
\mathrm{Tr}[\hat{cA}] &= c\mathrm{Tr}[\hat{A}], && (3.130)\\
\mathrm{Tr}[\hat{A}\otimes\hat{B}] &= \mathrm{Tr}[\hat{A}]\otimes\mathrm{Tr}[\hat{B}], && (3.131)\\
\mathrm{Tr}[\hat{A}\hat{B}\hat{C}\hat{D}] &= \mathrm{Tr}[\hat{B}\hat{C}\hat{D}\hat{A}] = \mathrm{Tr}[\hat{C}\hat{D}\hat{A}\hat{B}] = \mathrm{Tr}[\hat{D}\hat{A}\hat{B}\hat{C}]. && (3.132)
\end{aligned}$$

where $\hat{A}$ and $\hat{B}$ are matrices and c is a constant.

Mixed State

The mixed state is the probabilistic combination of two or more pure states. For the case of the mixture of state $|\psi\rangle$ with probability $\mathcal{P}_\psi$ and state $|\phi\rangle$ with probability $\mathcal{P}_\phi$, with $\mathcal{P}_\psi + \mathcal{P}_\phi = 1$,

$$\hat{\rho}_m = \mathcal{P}_\psi|\psi\rangle\langle\psi| + \mathcal{P}_\phi|\phi\rangle\langle\phi| \tag{3.133}$$

As an example, let us find the mixed state mentioned earlier: the realistic alternative to the $|\Phi^+\rangle$. It entails the product states each with a probability of 1/2

$$\hat{\rho}_{HH,VV} = \frac{1}{2}|H\rangle_1|H\rangle_2\langle H|_1\langle H|_2 + \frac{1}{2}|V\rangle_1|V\rangle_2\langle V|_1\langle V|_2 \tag{3.134}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.135}$$

Notice the difference in the density matrix between the mixed and entangled state. That difference will lead to a real measurable effect, shown below.

Probability

The probability that one state $|\psi\rangle$ be in another state $|\phi\rangle$. That is the absolute-value squared of the inner product is:

$$|\langle\psi|\phi\rangle|^2 = \mathrm{Tr}(\hat{\rho}_\psi\hat{\rho}_\phi) \tag{3.136}$$

Exercise 38 Prove the previous relation.

Expectation Value

The expectation value of an operator $\hat{A}$ when the system is in state $|\psi\rangle$ is

$$\langle\psi|\hat{A}|\psi\rangle = \mathrm{Tr}(\hat{A}\hat{\rho}_\psi) \tag{3.137}$$

Transformation

If the state $|\psi\rangle$ is processed by a device represented by the unitary operator say $\hat{R}$, becoming state $|\phi\rangle$, the state-vector equation is

$$|\phi\rangle = \hat{R}|\psi\rangle \tag{3.138}$$

The density matrix of the transformed state can be obtained from the density matrix of the initial state via

$$\hat{\rho}_\phi = \hat{R}\hat{\rho}_\psi \hat{R}^\dagger, \tag{3.139}$$

where $\hat{R}^\dagger$ is the adjoint of $\hat{R}$.

3.5.5 Bell Inequalities

Let us work out the correlations; joint detection probabilities when we project the state of the light with polarizers at angles θ_1 and θ_2. Rather than using projection operators, we calculate the probability that the initial state be in state

$$|\theta_1\rangle_1 |\theta_2\rangle_2 \tag{3.140}$$

The density matrix is a bit laborious

$$\hat{\rho}_{\theta_1,\theta_2} = |\theta_1\rangle_1 |\theta_2\rangle_2 \langle\theta_1|_1 \langle\theta_2|_2 = \tag{3.141}$$

$$= \begin{pmatrix} \cos^2\theta_1\cos^2\theta_2 & \frac{1}{2}\cos^2\theta_1\sin 2\theta_2 & \frac{1}{2}\sin 2\theta_1\cos^2\theta_2 & \frac{1}{4}\sin 2\theta_1\sin 2\theta_2 \\ \frac{1}{2}\cos^2\theta_1\sin 2\theta_2 & \cos^2\theta_1\sin^2\theta_2 & \frac{1}{4}\sin 2\theta_1\sin 2\theta_2 & \frac{1}{2}\sin 2\theta_1\sin^2\theta_2 \\ \frac{1}{2}\sin 2\theta_1\cos^2\theta_2 & \frac{1}{4}\sin 2\theta_1\sin 2\theta_2 & \sin^2\theta_1\cos^2\theta_2 & \frac{1}{2}\sin^2\theta_1\sin 2\theta_2 \\ \frac{1}{4}\sin 2\theta_1\sin 2\theta_2 & \frac{1}{2}\sin 2\theta_1\sin^2\theta_2 & \frac{1}{2}\sin^2\theta_1\sin 2\theta_2 & \sin^2\theta_1\sin^2\theta_2 \end{pmatrix}. \tag{3.142}$$

Then, after some more labor, we get

$$\mathrm{Tr}(\hat{\rho}_{\theta_1,\theta_2}\hat{\rho}_{\Phi^+}) = \frac{1}{2}\cos^2(\theta_1 - \theta_2) \tag{3.143}$$

In contrast, redoing the calculation for mixed states gives

$$\mathrm{Tr}(\hat{\rho}_{\theta_1,\theta_2}\hat{\rho}_m) = \frac{1}{2}(\sin^2\theta_1\sin^2\theta_2 + \cos^2\theta_1\cos^2\theta_2) \tag{3.144}$$

Exercise 39 Demonstrate the last equation (Not to worry: most of the work was Eq. 3.142).

Equations 3.143 and 3.144 are obviously different. Note that both predict a probability of 1/2 when $\theta_1 = \theta_2 = 0$, yet they differ dramatically when $\theta_1 = -\theta_2 = \pi/4$: entangled correlations give 0 whereas mixed correlations give 1/8. This is consistent with the previous discussion of examining the two possibilities in the (H, V) and (D, A) bases.

The discrepancy given above is a first step in distinguishing quantum states from local realistic states. In 1964 John Bell put forth a successful effort to distinguish quantum correlations from local realistic ones. They are in the form of inequalities that a "complete" local realistic quantum mechanics would satisfy if only certain "hidden variables" in the theory existed. Quantum mechanics without any local realistic corrections would violate the inequalities. This started a 3-decade-long quest for a definitive answer, with several versions of inequalities put forth with experimentally confirmed violations. By now it is fairly well accepted that quantum mechanics rules. One of the versions of Bell inequalities we present here goes by the Clauser-Horne-Shimony-Holt (CHSH) inequality, after the names of the authors.

We begin by defining a variable E that expresses the correlation between the polarizations of the two photons. If they are perfectly correlated (parallel, as in state $|\Phi^+\rangle$) then $E = 1$ and if they are perfectly anticorrelated (i.e., orthogonal, as in state $|\Psi^-\rangle$) then $E = -1$. If they are uncorrelated then $E = 0$. The expectation value of the correlation can be defined for arbitrary angles θ_1 and θ_2. At these angles E is

$$\begin{aligned} E(\theta_1, \theta_2) =& (+1)P(\theta_1, \theta_2) + (+1)P(\theta_1 + \frac{\pi}{2}, \theta_2 + \frac{\pi}{2}) + \\ & (-1)P(\theta_1, \theta_2 + \frac{\pi}{2}) + (-1)P(\theta_1 + \frac{\pi}{2}, \theta_2), \end{aligned} \tag{3.145}$$

or simply

$$E(\theta_1, \theta_2) = P(\theta_1, \theta_2) + P(\theta_1 + \frac{\pi}{2}, \theta_2 + \frac{\pi}{2}) - P(\theta_1, \theta_2 + \frac{\pi}{2}) - P(\theta_1 + \frac{\pi}{2}, \theta_2), \tag{3.146}$$

It can be shown that for state $|\Phi^+\rangle$ we have that the expectation value of the correlation is

$$E_{\text{ent}}(\theta_1, \theta_2) = \cos[2(\theta_1 - \theta_2)] \tag{3.147}$$

The perfect correlation of the state $|\Phi^+\rangle$ is manifested in Eq. 3.147 because $E(\theta, \theta) = 1$ regardless of θ. The mixed state gives a different correlation

$$E_{\text{mix}} = \cos(2\theta_1)\cos(2\theta_2). \tag{3.148}$$

Therefore, one can find situations where the two expectation values give different results.

States $|\Phi^-\rangle$ and $|\Psi^+\rangle$ have a degree of correlation $E(\theta, \theta)$ that depends on the basis. For state $|\Psi^-\rangle$ the correlation is

$$E(\theta_1, \theta_2) = -\cos[2(\theta_1 - \theta_2)]. \tag{3.149}$$

Therefore, for state $|\Psi^-\rangle$ the correlation is $E(\theta, \theta) = -1$ regardless of θ.

CHSH defined a variable S that depends on 4 angles, two for each photon: θ_1, θ_1', θ_2 and θ_2'. It is given by

$$S = E(\theta_1, \theta_2) - E(\theta_1, \theta_2') + E(\theta_1', \theta_2) + E(\theta_1', \theta_2'), \tag{3.150}$$

where θ_1 and θ_1' are two polarizer angles for photon 1, and similarly, θ_2 and θ_2' are two angles for photon 2. The inequality that a local realistic theory must satisfy is:

$$|S| \leq 2. \tag{3.151}$$

Entangled states measured at certain angles will violate this inequality. This is done with an apparatus similar to the one in Fig. **??**.

Exercise 40 Find S for $\theta_1 = -45°$, $\theta_1' = 0$, $\theta_2 = -\pi/8$ and $\theta_2' = \pi/8$ to show that the inequality of Eq. 3.151 is violated for state $|\Phi^+\rangle$.

Exercise 41 Show that the inequality of Eq. 3.151 with E_{mix} of Eq. 3.148 is satisfied when $\theta_1 = -45°$, $\theta_1' = 0$, $\theta_2 = -\pi/8$ and $\theta_2' = \pi/8$.

3.5.6 Quantum State Tomography

In the previous section we presented the conditions that prove that quantum mechanics and nature are not consistent with local realism. Thus, we can prepare a state that contains the essence of superposition in quantum mechanics. Fair enough, we prepare the state using a simple apparatus, as shown before. But then we should ask, aside from a Bell test, how do we know how good is the state that we produced? The appropriate step is to measure the state. Because it can have some mixture in it due to distinguishability of an imperfect apparatus, we need to measure the density matrix of the state. We can do this by projective measurements using the apparatus of Fig. 3.9.

The Measured Density Matrix

Earlier we mentioned how we project a state. This operation is given by Eq. 3.136. Suppose that we consider the projection of the unknown state with density matrix $\hat{\rho}$ onto state $|H\rangle_1|H\rangle_2$, it can be easily seen that it yields $\mathrm{Tr}(\hat{\rho}\hat{\rho}_{\mathrm{HH}}) = \rho_{11}$. Thus, if we send the photon pairs through two horizontal polarizers, we would get a number of counts that is proportional to ρ_{11}. It can easily be seen that the projections onto states $|H\rangle_1|V\rangle_2$, $|V\rangle_1|H\rangle_2$ and $|V\rangle_1|V\rangle_2$ directly yield respective measurements of ρ_{22}, ρ_{33} and ρ_{44}. Other state projections, such as $|D\rangle_1|R\rangle_2$, give rise to linear combinations of matrix elements ρ_{ij}. Thus, one can find a set of 16 projections, and by linear inversion obtain all the elements ρ_{ij}. A convenient and straight-forward method to do this has been described by James *et al.*[1] Their method uses the projections shown in Table 3.5.6. We also show the expected probabilities for

Table 3.1: Probabilities for the projections involved in the quantum state tomography of the state $|\Phi^+\rangle$ and $|\Psi^-\rangle$ and the corresponding mixed states.

State	Ψ^+ pure	(HH,VV) mixed	Φ^-	(HV,VH) mixed
$\|H\rangle_1\|H\rangle_2$	0.5	0.5	0	0
$\|H\rangle_1\|V\rangle_2$	0	0	0.5	0.5
$\|V\rangle_1\|V\rangle_2$	0.5	0.5	0	0
$\|V\rangle_1\|H\rangle_2$	0	0	0.5	0.5
$\|R\rangle_1\|H\rangle_2$	0.25	0.25	0.25	0.25
$\|R\rangle_1\|V\rangle_2$	0.25	0.25	0.25	0.25
$\|D\rangle_1\|V\rangle_2$	0.25	0.25	0.25	0.25
$\|D\rangle_1\|H\rangle_2$	0.25	0.25	0.25	0.25
$\|D\rangle_1\|R\rangle_2$	0.25	0.25	0.25	0.25
$\|D\rangle_1\|D\rangle_2$	0.5	0.25	0	0.25
$\|R\rangle_1\|D\rangle_2$	0.25	0.25	0.25	0.25
$\|H\rangle_1\|D\rangle_2$	0.25	0.25	0.25	0.25
$\|V\rangle_1\|D\rangle_2$	0.25	0.25	0.25	0.25
$\|V\rangle_1\|L\rangle_2$	0.25	0.25	0.25	0.25
$\|H\rangle_1\|L\rangle_2$	0.25	0.25	0.25	0.25
$\|R\rangle_1\|L\rangle_2$	0.5	0.25	0.5	0.25

[1] D.F.V. James, P.G. Kwiat, M.J. Munro and A.G. White "Measurement of qubits," *Physical Review A* **64**, 052312 (2001).

the pure and mixed states. Notice that only two projections give a difference between pure and mixed. Figure 3.11 shows the measured density matrices of actual data obtained when producing the states $|\Phi^+\rangle$, $|\Psi^-\rangle$ and the mixed state for (HH,VV). In principle this linear algebraic procedure is enough to

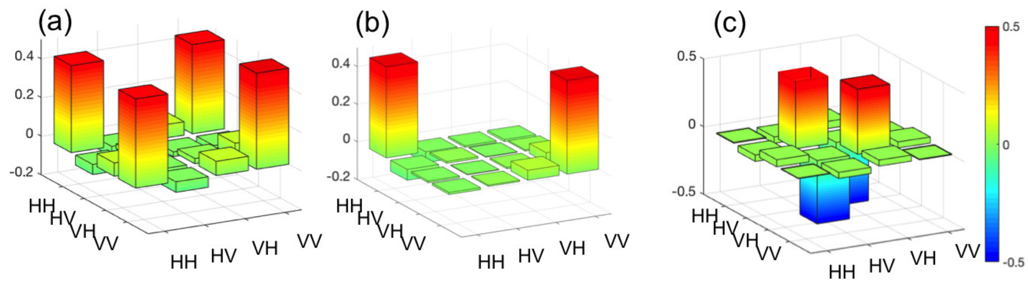

Figure 3.11: Quantum state tomography of two pure states: $|\Phi^+\rangle$ (a) and $|\Psi^-\rangle$ (c); and the mixed state corresponding to (HH,VV). Experimentally, state (a) was produced directly from the source described in Sec. 3.5.3; state (b) was produced in the same way but by adding a birefringent optic in the path of the pump that made the horizontal and vertical components of the pump incoherent, and thus producing an incoherent mixture of down converted photons in HH and VV; state (c) was produced with the setup of (a) but by adding two half-wave plates in the path of photon 2: one at $\pi/2$ with the horizontal to flip H and V and the other aligned with the vertical to flip the sign.

give us the proper density matrix, although there are ways to optimize the result even further that we will not discuss here.

Fidelity

What do we do with the measured density matrix? Since the density matrix can be represented by

$$\hat{\rho} = \sum_i p_i |\psi_i\rangle\langle\psi_i|, \tag{3.152}$$

where p_i are probabilities and ψ_i are pure states, we can reinterpret the equation as an expansion in terms of eigenvalues (p_i) and eigenvectors (ψ_i). If we diagonalize the state of Fig. 3.11(a), we obtain:

$$p_1 = 0, p_2 = 0, p_3 = 0.02, p_4 = 0.98.$$

This implies that most of the probability is in the fourth eigenvector:

$$|\psi_{a,4}\rangle = \begin{pmatrix} 0.68 \\ -0.06 \\ 0.07 \\ 0.73 \end{pmatrix}, \tag{3.153}$$

which is very close to $|\Phi^+\rangle = (0.71, 0, 0, 0.71)^T$. For the case of Fig. 3.11(c) we get similar results. The eigenvalues are

$$p_1 = 0, p_2 = 0, p_3 = 0.03, p_4 = 0.97.$$

The last eigenvalue corresponds to the eigenvector

$$|\psi_{c,4}\rangle = \begin{pmatrix} 0.07 \\ -0.71 \\ 0.69 \\ 0.07, \end{pmatrix}, \tag{3.154}$$

which is close to $|\psi^-\rangle = (0, -0.71, 0.71, 0)^T$.

Alternatively, we can calculate the fidelity of the state measured in (a) with state $|\Phi^+\rangle$. That is the probability of projecting of the measured state onto $|\Phi^+\rangle$. One possibility is to use Eq. 3.136. This however is restricted to the case of pure states. For states that are mixed the fidelity between two states $|\psi\rangle$ and $|\phi\rangle$ is given by

$$\mathcal{F} = \left[\mathrm{Tr}\left(\sqrt{\sqrt{\hat{\rho}_\phi}\hat{\rho}_\psi\sqrt{\hat{\rho}_\phi}}\right)\right]^2. \tag{3.155}$$

Applying the above relation for the fidelity of the density matrix of Fig. 3.11(a) with state $|\Phi^+\rangle$, we get the value listed in Table 3.2. Similarly, the fidelity of the state in (c) with state $|\Psi^-\rangle$ is given in the table. Note that there are not too dis-similar to the eigenvalues obtained in the diagonalization of the corresponding density matrices. The state of Fig. 3.11(b) is a mixture of two eigenstates: $|H\rangle_1|H\rangle_2$ and $|V\rangle_1|V\rangle_2$. The fidelity with these states are shown in Table 3.2, near 0.5, which is to be expected.

Concurrence and Tangle

Consider a 2-qubit state, such as the ones we have been discussing:

$$|\psi\rangle = a|H\rangle_1|H\rangle_2 + b|H\rangle_1|V\rangle_2 + c|V\rangle_1|H\rangle_2 + d|V\rangle_1|V\rangle_2 \tag{3.156}$$

Table 3.2: Quantum measures of the states given by the measured density matrices of Fig. 3.11.

State	Fidelity ($\mathcal{F}$)	Tangle (T)	Linear Entropy (S)
(a)	0.97	0.92 ± 0.05	0.05 ± 0.08
(b)	0.48, 0.50	0.009 ± 0.003	0.66 ± 0.02
(c)	0.96	0.96 ± 0.03	0.03 ± 0.04

If $b = c = 0$ and $a = d = 1/\sqrt{2}$ we are left with state $|\Phi^+\rangle$. The latter state is non-separable or non-factorable. Conversely, if $c = d = 0$ and $a = b = 1/\sqrt{2}$, the state is separable into a product state of photons 1 and 2.

Exercise 42 Show that if $ad = bc$ the state can be factored into a product of states of the two photons.

The concurrence for a pure state of two photons can be shown to be given by

$$C = 2|ad - bc| \geq 0. \tag{3.157}$$

Thus, it is zero for product states and 1 for maximally entangled states (i.e., states $|\Phi^{\pm}\rangle$ and $|\Psi^{\pm}\rangle$). For a general state that can have a certain degree of mixture the concurrence has a more elaborate definition that we do not give here, but it still remains as a way to quantify the entanglement or the non-separability. A fully mixed state is indeed not entangled, and so C=0. The Tangle is a more stringent measure and given by

$$T = C^2. \tag{3.158}$$

The measures of tangle for the the states of Fig. 3.11 are given in Table 3.2. Note that (a) and (c) are almost pure and entangled, whereas (b) has no entanglement whatsoever.

Linear Entropy

In information theory the entropy is the measure of the uncertainty in the state of the system. An associated quantity is the linear entropy. This quantity quantifies the degree of mixture in a quantum state. For a two-qubit system is given by

$$S = \frac{4}{3}\left(1 - \mathrm{Tr}[\hat{\rho}^2]\right) = \frac{4}{3}\left(1 - \sum_i p_i^2\right). \tag{3.159}$$

For a pure state there eigenvalues of $\hat{\rho}$ has probability 1 (the other 3 are zero), and thus $S = 0$. Alternatively, for a pure state: $\hat{\rho}^2 = \hat{\rho}$, so the linear entropy of Eq. 3.159 is zero. States (a) and (c) of Fig. 3.11 give linear entropies that are consistent with zero. Conversely, for a fully mixed state $S = 1$. A fully mixed state would only have diagonal elements with equal probability of 1/4, which gives $S = 1$ with Eq. 3.159. State (b) of Fig. 3.11 is a mixture of only two states with probabilities close to 1/2. Therefore, the linear entropy given in Table 3.2 is near 0.5.

Werner States

In a given situation where either the production of the state is imperfect or the time evolution interactions produce decohering effects, the quantum measures of the state are somewhat in between for both T and S. Werner states are a peculiar type of states that straddle between full entanglement and full mixture. The density matrix of a Werner state is given by

$$\hat{\rho}_W = p|\psi\rangle\langle\psi| + (1-p)\frac{\hat{I}}{4}, \tag{3.160}$$

where p is a parameter between 0 and 1 representing the probability of either aspect of the state, and $\hat{I}/4$ being the diagonal matrix representing the fully mixed state.

3.6 Stokes Parameters and the Mueller Matrix

The Stokes parameters specify the polarization properties of the light. Here we consider single and two-photon stokes parameters.

3.6.1 One Photon Qubit

We will be using the horizontal-vertical basis: Eqs. 3.11 and 3.12. The density matrix ρ corresponding to a state $|\psi\rangle$ can be expressed in terms of the Stokes parameters

$$\hat{\rho} = |\psi\rangle\langle\psi| = \frac{1}{2}\sum_{i=0}^{3} s_i\hat{\sigma}_i = \frac{1}{2}\begin{pmatrix} s_0 + s_1 & s_2 + is_3 \\ s_2 - is_3 & s_0 - s_1 \end{pmatrix}, \tag{3.161}$$

where the σ_i are the Pauli matrices, which we define here as:

$$\hat{\sigma}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.162}$$

$$\hat{\sigma}_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.163}$$

$$\hat{\sigma}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.164}$$

$$\hat{\sigma}_3 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \tag{3.165}$$

We note that we define the ordering of the Pauli matrices differently than usual because we use $|0\rangle = |H\rangle$ and $|1\rangle = |V\rangle$ instead of $|0\rangle = |R\rangle$ and $|1\rangle = |L\rangle$. The Stokes parameters can be obtained by measurements:

$$s_0 = |\langle H|\psi\rangle|^2 + |\langle V|\psi\rangle|^2 \tag{3.166}$$

$$s_1 = |\langle H|\psi\rangle|^2 - |\langle V|\psi\rangle|^2 \tag{3.167}$$

$$s_2 = |\langle D|\psi\rangle|^2 - |\langle A|\psi\rangle|^2 \tag{3.168}$$

$$s_3 = |\langle R|\psi\rangle|^2 - |\langle L|\psi\rangle|^2. \tag{3.169}$$

Conversely, in terms of the density matrix,

$$s_i = \mathrm{Tr}[\hat{\rho}\hat{\sigma}_i] \tag{3.170}$$

Example

Consider the diagonal state $|D\rangle$ defined by Eq. 3.20. The corresponding density matrix is

$$\hat{\rho}_D = |D\rangle\langle D| = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

It is easy to see that the corresponding Stokes parameters are

$$\begin{aligned}
s_0 &= |\langle H|D\rangle|^2 + |\langle V|D\rangle|^2 = \frac{1}{2} + \frac{1}{2} = 1 \\
s_1 &= |\langle H|D\rangle|^2 - |\langle V|D\rangle|^2 = \frac{1}{2} - \frac{1}{2} = 0 \\
s_2 &= |\langle D|D\rangle|^2 - |\langle A|D\rangle|^2 = 1 - 0 = 1 \\
s_3 &= |\langle R|D\rangle|^2 - |\langle L|D\rangle|^2 = \frac{1}{2} - \frac{1}{2} = 0.
\end{aligned}$$

Turning it around it can be seen that

$$\frac{1}{2}(\hat{\sigma}_0 + \hat{\sigma}_2) = \hat{\rho}_D$$

Exercise 43 Find the Stokes parameters for states $|V\rangle$ and $|R\rangle$

Exercise 44 Show that

$$\mathrm{Tr}[\hat{\sigma}_i \hat{\sigma}_j] = 2\delta_{ij}, \qquad (3.171)$$

where δ_{ij} is the Kronecker delta, equal to 1 if $i = j$, and 0 when $i \neq j$.

When a material or device changes the state of the light from $|\psi\rangle$ to $|\psi'\rangle$, its action can be represented by an operator $\hat{T}$ such that

$$|\psi'\rangle = \hat{T}|\psi\rangle. \qquad (3.172)$$

Equivalently, the final density matrix is given by

$$\hat{\rho}' = \hat{T}\hat{\rho}\hat{T}^\dagger \qquad (3.173)$$

The transformation is also explained using the Mueller matrix. If we consider the stokes parameters as elements of a vector $\vec{S}$

$$\vec{s} = \begin{pmatrix} s_0 \\ s_1 \\ s_2 \\ s_3 \end{pmatrix} \qquad (3.174)$$

such that

$$\vec{s'} = \hat{M}\vec{s} \qquad (3.175)$$

The final Stokes parameters can be expressed as

$$\begin{aligned} s'_i &= \mathrm{Tr}[\hat{\rho'}\hat{\sigma}_i] && (3.176) \\ &= \mathrm{Tr}[\hat{T}\hat{\rho}\hat{T}^\dagger\hat{\sigma}_i] && (3.177) \\ &= \mathrm{Tr}[\hat{T}(\sum_j s_j\hat{\sigma}_j)\hat{T}^\dagger\hat{\sigma}_i] && (3.178) \\ &= \sum_j \mathrm{Tr}[\hat{T}\hat{\sigma}_j\hat{T}^\dagger\hat{\sigma}_i]s_j && (3.179) \\ &= \sum_j M_{ij}s_j. && (3.180) \end{aligned}$$

Thus,

$$M_{ij} = \frac{1}{2}\text{Tr}[\hat{T}\hat{\sigma}_j\hat{T}^\dagger\hat{\sigma}_i]. \tag{3.181}$$

In using the cyclic trace property of Eq. 3.132 we put Eq. 3.181 in the more convenient form[2]

$$M_{ij} = \frac{1}{2}\text{Tr}[\hat{\sigma}_i\hat{T}\hat{\sigma}_j\hat{T}^\dagger]. \tag{3.182}$$

Example

In an earlier example we showed that $\hat{R}_{\pi/4}|H\rangle = |D\rangle$, then for this case $\hat{T} = \hat{R}_{\pi/4}$ and $\hat{T}^\dagger = \hat{R}_{-\pi/4}$. According to the previous relation we can find the elements of the Mueller matrix, which is a 4×4 matrix. It is easy to find:

$$\hat{M}(\hat{R}_{\pi/4}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Since

$$\vec{s}_H = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \vec{s}_D = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

we can see that $\hat{M}(\hat{R}_{\pi/4})\vec{s}_H = \vec{s}_D$.

A common method to get the Mueller matrix using linear algebra involves selecting a suitable set of 4 input states, which we label by k. If for a given input state k the Stokes parameter i is s_{ik}, with output Stokes parameter j given by s'_{jk}, then by definition of the Mueller matrix

$$s'_{jk} = M_{ji}s_{ik} \tag{3.183}$$

If we assemble the matrices N and N' formed by the Stokes vectors for states k such that

$$N_{ik} = s_{ik} \tag{3.184}$$

and

$$N'_{ik} = s'_{ik}, \tag{3.185}$$

[2]Toppel et al. New J. Phys. **16**, 073019 (2014).

then we get that

$$\hat{N}' = \hat{M}\hat{N}. \tag{3.186}$$

From this we get

$$\hat{M} = \hat{N}'\hat{N}^{-1} \tag{3.187}$$

provided $\det(\hat{N}) \neq 0$.

Example

Continuing the previous example we wish to determine the Mueller matrix by knowing what it does to 4 input states, suppose we chose our 4 k states to be $|H\rangle$, $|V\rangle$, $|D\rangle$ and $|R\rangle$. In a previous example we found $\vec{s}_H = (1\ 1\ 0\ 0)^T$ and $\vec{s}_D = (1\ 0\ 1\ 0)^T$. The answer to a follow-up exercise is $\vec{s}_V = (1\ -1\ 0\ 0)^T$ and $\vec{s}_R = (1\ 0\ 0\ 1)^T$. Since we "know the answer" (that the state changing device is a rotator set to $\pi/4$), we know that the output states are $\hat{R}_{\pi/4}|H\rangle = |D\rangle$, $\hat{R}_{\pi/4}|V\rangle = |A\rangle$, $\hat{R}_{\pi/4}|D\rangle = |V\rangle$ and $\hat{R}_{\pi/4}|R\rangle = \exp(\pi/4)|R\rangle$. The assembled matrices for this example are then

$$\hat{N} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \hat{N}' = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The inverse of $\hat{N}$ is

$$\hat{N}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix},$$

with $\det(\hat{N}) = -2$. It can be confirmed that $\hat{M}(\hat{R}_{\pi/4}) = \hat{N}'\hat{N}^{-1}$ with the matrices given above.

3.6.2 Two Photon Qubits

For two qubits the Stokes parameters have 16 elements $s_{\mu\nu}$ with $\mu, \nu = 0, 1, 2, 3$. The density matrix has 16 elements, as seen previously in Sec. 3.5.4. The two-qubit Stokes parameters are defined as:[3]

$$s_{\mu\nu} = \mathrm{Tr}[\hat{\rho}(\hat{\sigma}_\mu \otimes \hat{\sigma}_\nu)]. \tag{3.188}$$

[3]James et al. Phys. Rev. A **64**, 052312 (2001).

Conversely, the density matrix can be expressed in terms of the two-qubit Stokes parameters:

$$\hat{\rho} = \frac{1}{4}\sum_{\mu=0}^{3}\sum_{\nu=0}^{3} s_{\mu\nu}(\hat{\sigma}_\mu \otimes \hat{\sigma}_\nu) \tag{3.189}$$

Before we discuss the more general case, it is important to note that the two-photon density matrix already has the individual information of the state of each photon. The density matrix of photon 1 can be obtained by "tracing" over the second photon, yielding

$$\hat{\rho}_1 = \begin{pmatrix} \rho_{11} + \rho_{22} & \rho_{13} + \rho_{24} \\ \rho_{31} + \rho_{42} & \rho_{33} + \rho_{44} \end{pmatrix}. \tag{3.190}$$

This is equivalent to detecting the second photon without distinguishing its polarization. The Stokes vector of photon 1 independently of photon 2 is

$$\vec{s}_1 = \begin{pmatrix} s_{00} \\ s_{10} \\ s_{20} \\ s_{30} \end{pmatrix}, \tag{3.191}$$

and equivalently for photon 2:

$$\vec{s}_2 = \begin{pmatrix} s_{00} \\ s_{01} \\ s_{02} \\ s_{03} \end{pmatrix}. \tag{3.192}$$

In terms of the elements of the density matrix:[4]

$$\vec{s}_1 = \begin{pmatrix} \rho_{11} + \rho_{22} + \rho_{33} + \rho_{44} \\ \rho_{11} + \rho_{22} - \rho_{33} - \rho_{44} \\ 2\mathrm{Re}(\rho_{13}) + 2\mathrm{Re}(\rho_{24}) \\ 2\mathrm{Im}(\rho_{13}) + 2\mathrm{Im}(\rho_{24}) \end{pmatrix} \tag{3.193}$$

Bell states have a particular symmetry. It can be shown by direct calculation that

$$s_{\mu\nu} = 0 \quad \text{when} \quad \mu \neq \nu \tag{3.194}$$

[4]A. Abouraddy et al. Opt. Commun. **201**, 93-98 (2002).

for maximally entangled states. Then by the relations of Eqs. 3.191 and 3.192 we have $\vec{s}_1 = (1\ 0\ 0\ 0)^T$ and $\vec{s}_2 = (1\ 0\ 0\ 0)^T$. The degree of polarization for a single photon is given by

$$DOP = \sqrt{s_1^2 + s_2^2 + s_3^2}. \tag{3.195}$$

For a pure state $DOP = 1$. Thus, each photon in an entangled state has $DOP = 0$, and so it is individually unpolarized.

Another situation that may arise is the evolution of the density matrix when state-changing operation applies to only one photon. That is, when the state $|\psi\rangle$ is transformed by operation $\hat{T}$ on say photon 1, such that

$$|\psi'\rangle = (\hat{T} \otimes I_2)|\psi\rangle, \tag{3.196}$$

where I_2 is the two-dimensional identity matrix. Equivalently, the density matrix will transform

$$\hat{\rho}' = (\hat{T} \otimes I_2)\rho(\hat{T} \otimes I_2)^\dagger. \tag{3.197}$$

Replacing Eq. 3.189 into Eq. 3.197 we get

$$\hat{\rho}' = \frac{1}{4}\sum_{\mu=0}^{3}\sum_{\nu=0}^{3} s_{\mu\nu}(\hat{T} \otimes I_2)(\hat{\sigma}_\mu \otimes \hat{\sigma}_\nu)(\hat{T} \otimes I_2)^\dagger, \tag{3.198}$$

$$= \frac{1}{4}\sum_{\mu=0}^{3}\sum_{\nu=0}^{3} s_{\mu\nu}(\hat{T}\hat{\sigma}_\mu\hat{T}^\dagger) \otimes \hat{\sigma}_\nu, \tag{3.199}$$

For Bell states (and consequently the relation 3.194) the latter equation becomes

$$\hat{\rho}' = \frac{1}{4}\sum_{\mu=0}^{3} s_{\mu\mu}(\hat{T}\hat{\sigma}_\mu\hat{T}^\dagger) \otimes \hat{\sigma}_\mu. \tag{3.200}$$

If we now replace $\hat{\rho}'$ into Eq. eq:stokes2 to obtain the Stokes parameters of the transformed state, we get

$$s'_{ij} = \mathrm{Tr}[\hat{\rho}'(\hat{\sigma}_i \otimes \hat{\sigma}_j)], \tag{3.201}$$

$$= \frac{1}{4}\sum_{\mu=0}^{3} s_{\mu\mu}\mathrm{Tr}[((\hat{T}\hat{\sigma}_\mu\hat{T}^\dagger) \otimes \hat{\sigma}_\mu)(\hat{\sigma}_i \otimes \hat{\sigma}_j)] \tag{3.202}$$

$$= \frac{1}{4}\sum_{\mu=0}^{3} s_{\mu\mu}\mathrm{Tr}[(\hat{T}\hat{\sigma}_\mu\hat{T}^\dagger\hat{\sigma}_i) \otimes (\hat{\sigma}_\mu\hat{\sigma}_j)] \tag{3.203}$$

Using trace property Eq. 3.131 in the latter equation yields

$$s'_{ij} = \frac{1}{4}\sum_{\mu=0}^{3} s_{\mu\mu}\mathrm{Tr}[\hat{T}\hat{\sigma}_\mu\hat{T}^\dagger\hat{\sigma}_i] \otimes \mathrm{Tr}[\hat{\sigma}_\mu\hat{\sigma}_j] \tag{3.204}$$

Using the cyclic property of the trace (Eq. 3.132) and the property of Pauli matrices (Eq. 3.171), we get

$$s'_{ij} = \frac{1}{4}s_{jj}\mathrm{Tr}[\hat{\sigma}_i\hat{T}\hat{\sigma}_j\hat{T}^\dagger] \tag{3.205}$$

$$= s_{jj}M_{ij} \tag{3.206}$$

where we used Eq. 3.182 in the last equation to replace in the Mueller matrix element of the transformation of photon 1. Thus

$$M_{ij} = \frac{s'_{ij}}{s_{jj}}. \tag{3.207}$$

The meaning of this result is that we can get all Mueller matrix elements from the two photon density matrix when we use a Bell state as the input state. This is a form of non-local Mueller polarimetry.

The values of the non-zero Stokes parameters for each Bell state are shown in Table 3.3. The degree of two-photon polarization must then be the degree

Table 3.3: Values of the non-zero 2-qubit Stokes parameters for each Bell state.

Non-zero	Φ^+	Φ^-	Ψ^+	Ψ^-
s_{00}	1	1	1	1
s_{11}	1	1	-1	-1
s_{22}	1	-1	1	-1
s_{33}	-1	1	1	-1

to which the polarizations of the two photons is correlated. It is given by

$$DOP_2 = \frac{1}{2}\left(\sum_{i,j=1}^{3} s_{ij}^2 - 1\right) \tag{3.208}$$

Thus, it can be seen that $DOP_2 = 1$ for all maximally entangled Bell states. It can be seen that it is zero for any separable state.

Exercise 45 The Mueller matrix for a 90-degree rotator is given by

$$M(R_{\pi/2}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

A rotator placed in the path of a photon converts state Φ^+ to Ψ^-. Using Table 3.3 and Eq. 3.207 show that the Mueller matrix is the one shown above.

3.7 Two Photons in Momentum Space

3.7.1 Hong-Ou-Mandel Interference

Consider two photons created by spontaneous parametric down conversion are incident on a beam splitter. Photons 1 and 2 arrive at the beam splitter.

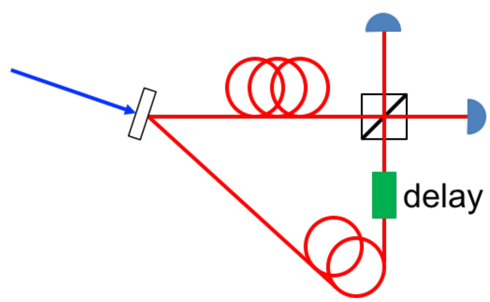

Figure 3.12: HOM experiment setup. Two photons produced by spontaneous parametric down-conversion are redirected by fibers to a beam splitter. One of the photons can be delayed or advanced relative to the other one. Detectors measure the coincident events after the beam splitter.

The wavefunction for the photons as bosons has to be symmetric to the exchange of photon labels. Thus, the initial state of the light is given by

$$|\psi\rangle_i = \frac{1}{\sqrt{2}} \left(|x\rangle_1 |y\rangle_2 + |y\rangle_1 |x\rangle_2 \right) \tag{3.209}$$

The matrix form of the two photon states in the momentum basis is:

$$|x\rangle_1|y\rangle_2 = \begin{pmatrix}1\\0\end{pmatrix}_1 \otimes \begin{pmatrix}0\\1\end{pmatrix}_2 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \tag{3.210}$$

and

$$|y\rangle_1|x\rangle_2 = \begin{pmatrix}0\\1\end{pmatrix}_1 \otimes \begin{pmatrix}1\\0\end{pmatrix}_2 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \tag{3.211}$$

The initial state is

$$|\psi\rangle_i = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}. \tag{3.212}$$

The matrix for the beam splitter is

$$\hat{B} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & i & i & -1\\ i & 1 & -1 & i\\ i & -1 & 1 & i\\ -1 & i & i & 1\end{pmatrix} \tag{3.213}$$

The final state after the beam spitter is

$$|\psi\rangle_f = \hat{B}|\psi\rangle_i = \frac{i}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix} \tag{3.214}$$

This final state involves the superposition of both photons going in the x-direction with both photons going in the y-direction. Thus, there are no coincidences recorded. This interference is due to both photons being indistinguishable.

Let us analyze the situation where both photons are partially distinguishable and partially indistinguishable. The density matrix representation of the photons in the initial state when both are indistinguishable is

$$\hat{\rho}_{\text{ind}} = |\psi\rangle\langle\psi| = \frac{1}{2}\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 1 & 1 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}. \tag{3.215}$$

When both photons are distinguishable, then they are in a mixed state. The two possibilities that are distinguishable are given by their respective density matrices:

$$\hat{\rho}_{\rm xy} = |x\rangle_1 |y\rangle_2 \langle y|_2 \langle x|_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.216}$$

and

$$\hat{\rho}_{\rm yx} = |y\rangle_1 |x\rangle_2 \langle x|_2 \langle y|_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.217}$$

The mixed state is defined as

$$\hat{\rho}_{\rm dis} = \frac{1}{2}\hat{\rho}_{\rm xy} + \frac{1}{2}\hat{\rho}_{\rm yx} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.218}$$

where the factors of 1/2 refer to the probability of occurrence of those possibilities.

We can set up a general state that is partially distinguishable state as

$$\hat{\rho}_g = p\hat{\rho}_{ind} + (1-p)\hat{\rho}_{\rm dis} \tag{3.219}$$

where p is the probability that the photons are indistinguishable. The density matrix of the final state is

$$\hat{\rho}_f = \hat{B}\hat{\rho}_i\hat{B}^+. \tag{3.220}$$

One can use this last equation to verify the two extreme outcomes:

$$\hat{\rho}_{f-\rm ind} = \hat{B}\hat{\rho}_{i-\rm ind}\hat{B}^+ = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \tag{3.221}$$

where the outcome is consistent with $|\psi\rangle_f \langle\psi|_f$ of Eq. 3.214; and

$$\hat{\rho}_{f-\rm dis} = \hat{B}\hat{\rho}_{i-\rm dis}\hat{B}^+ = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}. \tag{3.222}$$

The final state for the general case is

$$\hat{\rho}_{gf} = \frac{1}{4}\begin{pmatrix} 1+p & 0 & 0 & 1+p \\ 0 & 1-p & -1+p & 0 \\ 0 & -1+p & 1-p & 0 \\ 1+p & 0 & 0 & 1+p \end{pmatrix} \tag{3.223}$$

The probability of detecting coincidences is

$$P = \mathrm{Tr}[\hat{\rho}_{gf}\hat{\rho}_{xy}] + Tr[\hat{\rho}_{gf}\hat{\rho}_{yx}] \tag{3.224}$$

Figure 3.13 shows the coincidence probability as a function of p. If we model

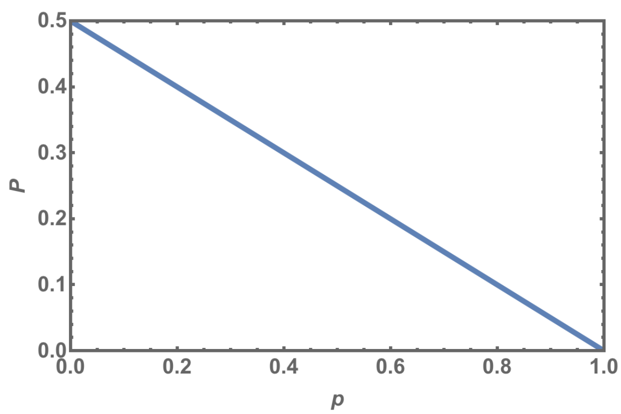

Figure 3.13: Calculated probability of measuring a coincidence (Eq. 3.224) as a function of p.

the probability p by the overlap of the photon wavepacket of coherence length l_c, then a graph of coincidences as a function of the delay or shift in the overlap in the amplitudes of the two photons, is shown in Fig. 3.14. The photons can be thought of as given by an amplitude that can be modeled by a Gaussian of width given by the coherence length of the light. The coherence length is related to the bandwidth $\delta\lambda$, where λ is the center wavelength of the photon, by

$$l_c = \frac{\lambda^2}{\delta\lambda} \tag{3.225}$$

Thus delaying one photon relative to the other one produces a difference in the overlap, yielding full interference (indistinguishability) for zero overlap, and no interference (distinguishability) when the delay/advance is greater

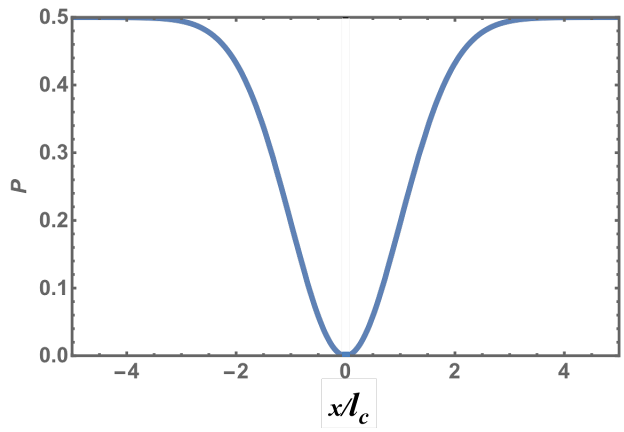

Figure 3.14: Calculated probability of measuring a coincidence (Eq. 3.224) as a function of the delay in (overlap) in the photon amplitudes x. The photons' coherence length is l_c.

than the coherence length. A scan of the delay gives rise to the Hong-Ou-Mandel "dip" in the coincidence signal of the two detectors, as shown in Fig. 3.14.

More Degrees of Freedom: Photon-Momentum-Polarization

In the HOM case we have two photons and two momentum states. We can add the polarization degree of freedom as well. In the previous version, the HOM interference is decided by the temporal distinguishability of the two photons due to distinct arrival times of the photons to the beam splitter. We can keep the temporal indistinguishability and make polarization to be the distinguishing parameter. We do this by adding waveplates before the beam splitter.

In the case of Fig. 3.12, one detail that was not mentioned is that the fibers have to preserve the polarization, so in practice we use polarization-maintaining fibers. We then place half-wave plates in front of the fiber entrance. Each photon now has polarization and momentum degrees of freedom. Foe example the state for the two photons incident to the beam splitter with one with horizontal polarization traveling along x with the other photon with polarization oriented an angle θ traveling along y is:

$$|\psi\rangle_i = \frac{1}{\sqrt{2}} \left(|x, H\rangle_1 |y, \theta\rangle_2 + |y, \theta\rangle_1 |x, H\rangle_2 \right) \tag{3.226}$$

The beam splitter operator is now

$$\hat{B}_3 = \hat{B} \otimes \hat{I} \otimes \hat{I}, \tag{3.227}$$

where the second and third spaces are the polarization and photon modes. Doing calculations by hand in this problem becomes laborious. Using computational tools eases the effort. The final state is obtained by applying

$$|\psi\rangle_f = \hat{B}_3 |\psi\rangle_i. \tag{3.228}$$

The probability for obtaining coincidences, is obtained by

$$P_{xy} = \sum_{i=H,V} \sum_{j=H,V} \left(|\langle x, i|_1 \langle y, i|_2 \psi\rangle_f|^2 + |\langle y, i|_1 \langle x, j|_2 \psi\rangle_f|^2 \right). \tag{3.229}$$

Interestingly, this elaborate calculation results in a simple answer:

$$P_{xy} = \frac{1}{2} \sin^2 \theta, \tag{3.230}$$

but it only underscores that distinguishability is the key to HOM interference.

3.7.2 Biphoton Interference

3.8 Continuous Variables: Time and Energy

Consider now the situation where two photons are produced in a setup such as that of Fig. 3.15 An incident photon produces a pair of photons with energy that add to the energy of the incident photon. They are in an entangled state of energy

$$|\psi\rangle = \int A(E) |E\rangle_1 |E_0 - E\rangle_2 dE, \tag{3.231}$$

where $A(E)$ is the amplitude of producing a given pair of energies $(E, E_0 - E)$, and where E_0 is the energy of the pump beam. This is an energy wavepacket.

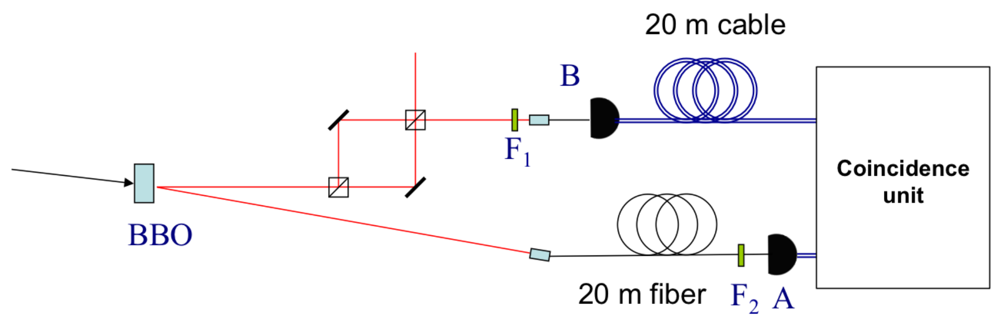

Figure 3.15: Experiment involving delayed choice.

Suppose that the uncertainty in the energy is δE. From the time-energy uncertainty relation we have

$$\delta t \sim \frac{h}{\delta E}, \tag{3.232}$$

which we can interpret as the temporal width of the wavepacket. Thus, a large energy uncertainty implies a short temporal width, and conversely, a small energy uncertainty implies a long temporal width of the wavepacket. We can translate the temporal width to an actual distance

$$\ell_c = c\delta t. \tag{3.233}$$

This distance is also known as the coherence length of the light.

We now send one of the photons to an interferometer, while the other eventually goes straight to a detector. We put filters F_1 and F_2 before sending them to the detectors. The interferometer can be represented by the operator $\hat{U}$ acting on photon 1. If the coefficients of reflection and transmission of the beam splitters are r and t, and if the path-length difference of the arms of the interferometer is x, then the after the interferometer the state of the photons is (un-normalized):

$$|\psi'\rangle = \hat{U}|\psi\rangle = \int A(E)rt(1 + e^{\mathrm{i}2\pi Ex/hc})|E\rangle_1|E_0 - E\rangle_2 dE. \tag{3.234}$$

We then pass the photons through two filters with transmission functions $a_1(E)$ and $a_2(E)$. Detectors following the filters complete an energy-projective measurement into states

$$|\psi'\rangle_i = \int a_i(E')|E'\rangle_i dE' \tag{3.235}$$

$i = 1, 2$. The detection probability is then given by

$$P = \left|\langle\psi'|_1\langle\psi''|_2|\psi\rangle\right|^2 . \tag{3.236}$$

Replacing the expressions for the states that are detected gives

$$P = \int\int\int dE dE' dE'' a_1(E') a_2(E'') A(E) \left|\langle E'|_1\langle E''|_2 \hat{U}|E\rangle_1 |E_0 - E\rangle_2\right|^2 . \tag{3.237}$$

Note that there is no time dependence. The final probability is independent of when the measurements are performed. This is similar to the famous "spooky action at a distance" situation with polarization-entangled photons: the outcome of the correlations is independent of the order in which the measurements is made. The integral over the individual energy summations are reduced due to the orthogonality of energy eigenstates

$$\langle E'|E\rangle = \delta(E' - E), \tag{3.238}$$

and

$$\langle E''|E_0 - E\rangle_2 = \delta(E'' - E_0 + E), \tag{3.239}$$

giving rise to the following probability:

$$P = \int a(E) \left|rt(1 + e^{i2\pi Ex/(hc)})\right|^2 dE, \tag{3.240}$$

where $a(E) = A(E)a_1(E)a_2(E_0 - E)$. If we make, for simplicity that $a(E) = 1/(E_2 - E_1)$ when $E_1 \leq E \leq E_2$, and 0 otherwise then the integral is straightforward to solve giving

$$P = 2RT\left(1 + \frac{2hc}{\pi\Delta Ex}\sin\frac{\pi\Delta Ex}{hc}\cos\frac{2\pi\overline{E}x}{hc}\right) \tag{3.241}$$

where $\Delta E = E_2 - E_1$ and $\overline{E} = (E_1 + E_2)/2$. The above equation simplifies to

$$P = \frac{1}{2}\left(1 + \frac{\sin\alpha}{\alpha}\sin\frac{2\pi x}{\overline{\lambda}}\right), \tag{3.242}$$

where $\alpha = \pi\Delta Ex/(hc)$ and $rr^*tt^* = 1/4$. Note also that

$$\alpha = \frac{\pi x}{\ell_c} \tag{3.243}$$

Thus, when $x \ll \ell_c$ we have $\sin\alpha/\alpha \to 1$, so the probability is

$$P = \frac{1}{2}\left[1 + \sin\left(\frac{2\pi x}{\overline{\lambda}}\right)\right], \tag{3.244}$$

with $\overline{\lambda} = hc/\overline{E}$, which corresponds to interference that varies with the path length difference x. As x increases to the limit when $x \gg \ell_c$, or $\alpha \to \infty$, then $\sin\alpha/\alpha \to 0$, the interference disappears and the probability becomes $P = 1/2$.

The setup of the apparatus of Fig. 3.15 has an interesting twist. First, the light going to the two detectors has filters F_1 and F_2, one for each photon. The final bandwidth of the light is determined by the product of the transmission curves of the two filters. Because the photons are detected in coincidence, the bandwidth is effectively determined by the filter with the narrower bandwidth. When $x \gg \ell_c$ the path-length difference is much larger than the length of the wavepacket, and so the wavepackets do not overlap in time when the light exits the interferometer. In the data of Fig. 3.16 the length of the interferometer, $x = \ell + L$, where $L = 81$ μm is fixed and ℓ is scanned over a few wavelengths. In Fig. 3.16(a) the bandwidth of the filter system is 40 nm resulting in a coherence length $\ell_c = 16$ nm. One could also say that in Fig. 3.16(a) the paths of the interferometer are distinguishable because a timing experiment that would reveal the path taken by the light (see insert in the figure). Conversely, in Fig. 3.16(b) the bandwidth of the filter system is 1 nm resulting in a coherence length $\ell_c = 640$ μm. This is the limit when $x \ll \ell_c$, where the probability amplitude wavepackets of the photon going through the interferometer overlap (see insert). As consequence, the paths are indistinguishable, and so there is interference. Thus, the selection of the filter(s) determines whether the light interferes or not. One can say that case (a) displays the particle aspect of the light, and case (b) displays the wave aspect.

In the experiment one photon goes to a filter and detector immediately after going through the interferometer, while the other one continues through a long optical fiber to finally reach a filter and detector. To make this point more dramatic, in the setup we locate the bandwidth-determining filter after the long fiber, and thus delaying the filtering action. However, when the delayed photon goes through the filter, the other photon going the interferometer has already been detected and no longer exists! Often when discussing entangled photons, it is often said that detecting one photon collapses the

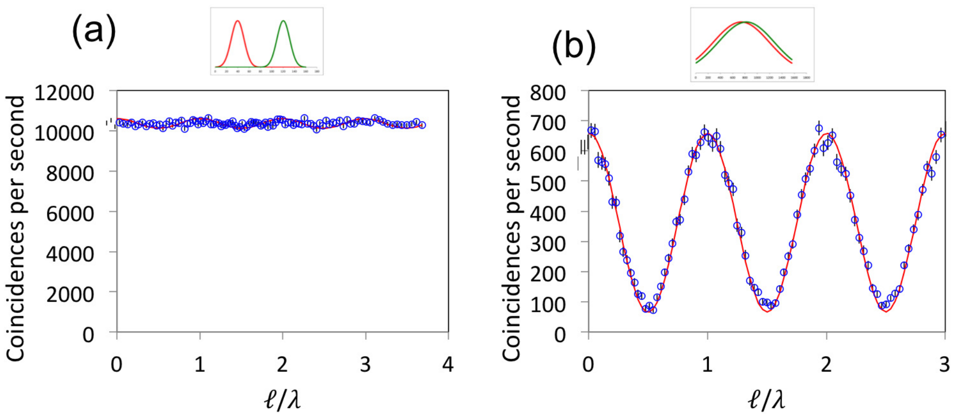

Figure 3.16: Measurements of single-photon interference for a difference in path length of about 81 μm. In (a) the paths are distinguishable and in (b) the paths are indistinguishable. The Inserts show schematically the length of the wavepackets for each case: 16 μm for (a) and 640 μm for (b), as they exit the interferometer.

wavefunction, instantaneously projecting the state of the other photon. This problem leads one to think that the delayed photon controls the bandwidth of the photon going through the interferometer, thus constituting a back action into the past. However, that is not so. Once the first photon is detected, that detection projects the second photon into a state of the carries *both* the wave and particle information. The delayed detection then projects the state further selecting what type of information, wave-like or particle-like, is made available.